Finite Element Method(FEM) for Two Dimensional Laplace Equation with Dirichlet Boundary Conditions

April 9, 2007

1 Variational Formulation of the Laplace Equation

The problem is to solve the Laplace equation

$$\nabla^2 u = 0 \tag{1}$$

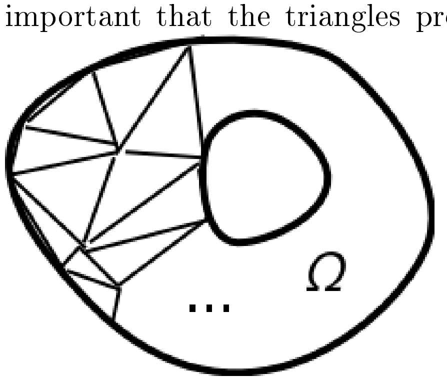
in domain Ω subject to Dirichlet boundary conditions on $\partial\Omega$. We know from our study of the uniqueness of the solution of the Laplace equation that finding the solution is equivalent to finding u that minimizes

$$W = \frac{1}{2}\int_{\Omega} ||\nabla u||^2 d\tau \tag{2}$$

subject to the same boundary conditions. Here the differential $d\tau$ denotes the volume differential and stands for $dxdy$ for a plane region. W has interpretations such as stored energy or dissipated power in various problems.

2 Meshing

First we approximate the boundary of Ω by polygons. Then Ω can be divided into small triangles called triangular elements. There is a great deal of flexibility in this division process. The term *meshing* is used for this division. For the resulting FEM matrices to be well-conditioned it is important that the triangles produced by meshing should not have angles which are too small.



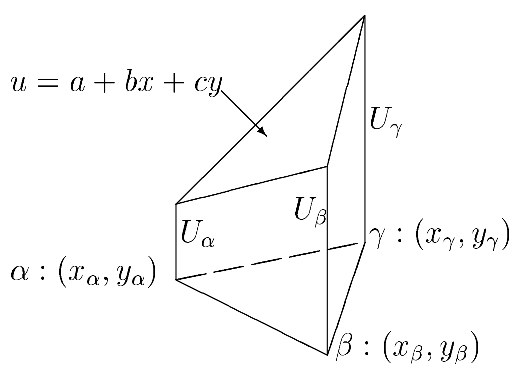
At the end of the meshing process the following quantities are created.

- N_v: number of vertices or nodes.
- $N_v \times 2$ array of real numbers holding the x and y coordinates of the vertices.
- N_e: number of triangular elements.

- $N_e \times 3$ array of integers holding the vertices of the triangular elements.
- N_{vf}: Number of vertices on which the u values are not specified or *free.*
- $N_{vf} \times 1$ array of integers holding free vertex indices.
- N_{vp}: Number of vertices on which the u values are specified or *prescribed.*
- $N_{vp} \times 1$ array of integers holding prescribed vertex indices.
- $N_{vp} \times 1$ array of real numbers holding prescribed u values.

Data structures holding adjacency information for the vertices, edges, and the triangular elements are also generated by sophisticated FEM meshing subroutines.

3 Planar Approximation over a Triangle



Let us consider a triangular element whose node numbers are α, β, and γ. The node coordinates are (x_α, y_α), (x_β, y_β), (x_γ, y_γ). On this triangle $u(x,y)$ is assumed to have a planar variation:

$$u(x,y) = a + bx + cy = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{3}$$

First a, b, c are to be determined in terms of the values of $u(x,y)$ at the nodes U_α, U_β, U_γ, using the three equations

$$u(x_j, y_j) = U_j, \qquad j \in \{\alpha, \beta, \gamma\} \tag{4}$$

In matrix form the equations are

$$\begin{bmatrix} 1 & x_\alpha & y_\alpha \\ 1 & x_\beta & y_\beta \\ 1 & x_\gamma & y_\gamma \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} U_\alpha \\ U_\beta \\ U_\gamma \end{bmatrix} \tag{5}$$

So

$$\begin{aligned} u(x,y) &= \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} 1 & x_\alpha & y_\alpha \\ 1 & x_\beta & y_\beta \\ 1 & x_\gamma & y_\gamma \end{bmatrix}^{-1} \begin{bmatrix} U_\alpha \\ U_\beta \\ U_\gamma \end{bmatrix} \\ &= \begin{bmatrix} 1 & x & y \end{bmatrix} \frac{1}{2A} \begin{bmatrix} x_\beta y_\gamma - x_\gamma y_\beta & x_\gamma y_\alpha - x_\alpha y_\gamma & x_\alpha y_\beta - x_\beta y_\alpha \\ y_\beta - y_\gamma & y_\gamma - y_\alpha & y_\alpha - y_\beta \\ x_\gamma - x_\beta & x_\alpha - x_\gamma & x_\beta - x_\alpha \end{bmatrix} \begin{bmatrix} U_\alpha \\ U_\beta \\ U_\gamma \end{bmatrix} \\ &= U_\alpha \psi_\alpha(x,y) + U_\beta \psi_\beta(x,y) + U_\gamma \psi_\gamma(x,y) \end{aligned} \tag{6}$$

where

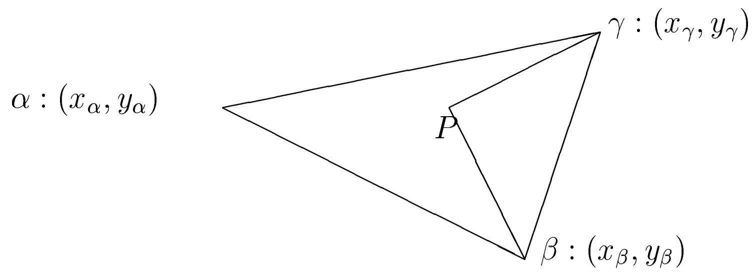
$$2A = x_\beta y_\gamma - x_\gamma y_\beta + x_\gamma y_\alpha - x_\alpha y_\gamma + x_\alpha y_\beta - x_\beta y_\alpha \tag{7}$$

and

$$\psi_\alpha(x, y) = \frac{1}{2A}[x_\beta y_\gamma - x_\gamma y_\beta + (y_\beta - y_\gamma)x + (x_\gamma - x_\beta)y] \tag{8}$$

etc. $2A$ is twice the area of the triangle $\alpha\beta\gamma$. The functions ψ_j are interpolatory in nature.

$$\psi_j(x_k, y_k) = \delta_{jk}, \quad j, k \in \{\alpha, \beta, \gamma\} \tag{9}$$



The ψ functions are also known as *natural* coordinates, *simplex* coordinates or *areal* coordinates. Let P be the point (x, y). Then

$$\psi_\alpha(x, y) = \frac{\text{Area of triangle } P\beta\gamma}{\text{Area of triangle } \alpha\beta\gamma} \tag{10}$$

Exercises:

- Show that $\psi_\alpha(x, y) + \psi_\beta(x, y) + \psi_\gamma(x, y) = 1$.
- At which point is $\psi_\alpha^2(x, y) + \psi_\beta^2(x, y) + \psi_\gamma^2(x, y)$ minimum? What is the minimum value?

Since the original domain is now approximated as a union of small triangular elements, the total W corresponding to stored energy or power dissipation can be expressed as a sum of element W's. For the $\alpha\beta\gamma$ element

$$W^{(e)} = \frac{1}{2}\int_{(\alpha\beta\gamma)} ||\nabla u||^2 d\tau \tag{11}$$

Now

$$\nabla u = U_\alpha \nabla\psi_\alpha + U_\beta \nabla\psi_\beta + U_\gamma \nabla\psi_\gamma \tag{12}$$

It should be noted that due to the planar variation of $u(x, y)$

$$\nabla\psi_\alpha(x, y) = \frac{1}{2A}[(y_\beta - y_\gamma)\hat{\mathbf{x}} + (x_\gamma - x_\beta)\hat{\mathbf{y}}] \tag{13}$$

etc. are constant over the entire element. So

$$W^{(e)} = \frac{1}{2}\int_{(\alpha\beta\gamma)} ||\nabla u||^2 d\tau = \frac{1}{2}\sum_{j=\alpha,\beta,\gamma}\sum_{k=\alpha,\beta,\gamma} U_j S_{jk}^{(e)} U_k \tag{14}$$

where

$$S_{jk}^{(e)} = \int_{(\alpha\beta\gamma)} (\nabla\psi_j)\cdot(\nabla\psi_k) d\tau, \quad j, k \in \{\alpha, \beta, \gamma\} \tag{15}$$

Let

$$U^{(e)} = \begin{bmatrix} U_\alpha \\ U_\beta \\ U_\gamma \end{bmatrix} \quad (16)$$

Then

$$W^{(e)} = \frac{1}{2} U^{(e)^T} S^{(e)} U^{(e)} \quad (17)$$

$S^{(e)}$ is called the element *Dirichlet* matrix. Show that,

$$S^{(e)}_{\alpha\beta} = \frac{1}{2A} [(y_\beta - y_\gamma)(y_\gamma - y_\alpha) + (x_\gamma - x_\beta)(x_\alpha - x_\gamma)] \quad (18)$$

etc. The Dirichlet matrix is symmetric, positive semi-definite. The expression for element energy is a quadratic form in the nodal values U_α, U_β, U_γ. Since W is expressed as a sum of all the element energies, it follows that W is given by a positive semi-definite quadratic form in the nodal values $U_1, U_2, \ldots U_N$.

$$W = \frac{1}{2} U^T S U \quad (19)$$

where

$$U = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_N \end{bmatrix} \quad (20)$$

and the Dirichlet matrix S has contributions from all element $S^{(e)}$'s. In the program developed here, S is initialized to zeros and as one loops over all elements any $S^{(e)}_{\alpha\beta}$ contribution is added to $S_{\alpha\beta}$. This is in contrast to other FEM programs which actually generate 3×3 element matrices and then combine all of them during *element assembly*.

4 Minimization of W

P of the N U values are *prescribed*. $N - P$ are free to vary. For simplicity of presentation we assume that the prescribed values come after the free values. (MATLAB and modern matrix libraries allow sub-matrices to be selected based on arbitrary index sets. So there is no need to actually number the prescribed nodes after the free nodes.) Then the matrices U and S may be partitioned as follows:

$$U = \begin{bmatrix} U_f \\ U_p \end{bmatrix} \quad (21)$$

$$S = \begin{bmatrix} S_{ff} & S_{fp} \\ S_{pf} & S_{pp} \end{bmatrix} \quad (22)$$

Note that due to the symmetry of S, $S_{fp} = S^T_{pf}$. In terms of the partitioned matrices W may be expressed as

$$W = \frac{1}{2} U^T S U \quad (23)$$

$$= \frac{1}{2} \begin{bmatrix} U^T_f & U^T_p \end{bmatrix} \begin{bmatrix} S_{ff} & S_{fp} \\ S_{pf} & S_{pp} \end{bmatrix} \begin{bmatrix} U_f \\ U_p \end{bmatrix} \quad (24)$$

$$= \frac{1}{2} U^T_f S_{ff} U_f + \frac{1}{2} U^T_p S_{pf} U_f + \frac{1}{2} U^T_f S_{fp} U_p + \frac{1}{2} U^T_p S_{pp} U_p \quad (25)$$

But since $U_p^T S_{pf} U_f$ is a scalar it equals its own transpose.

$$U_p^T S_{pf} U_f = (U_p^T S_{pf} U_f)^T = U_f^T S_{pf}^T U_p = U_f^T S_{fp} U_p \tag{26}$$

The last equality is due to the fact that $S_{fp} = S_{pf}^T$. So by combining the two equal middle terms of (25) we get

$$W = \frac{1}{2} U_f^T S_{ff} U_f + U_f^T S_{fp} U_p + \frac{1}{2} U_p^T S_{pp} U_p \tag{27}$$

Now this W is to be minimized with respect to U_f. At the minimum the gradient of W with respect to U_f equals zero.

$$\nabla_{U_f} W = S_{ff} U_f + S_{fp} U_p = 0 \tag{28}$$

So the solution is

$$U_f = -S_{ff}^{-1} S_{fp} U_p \tag{29}$$

Once U_f is known, U is known, and one computes $W = \frac{1}{2} U^T S U$. In physical problems, quantities related to W are usually important.

In eigenvalue problems like the Helmholtz equation, and loading problems like the Poisson equation, not only $\frac{1}{2} \int_\Omega ||\nabla u||^2 d\tau$, but also $\frac{1}{2} \int_\Omega u^2 d\tau$ is of importance. Just as $\frac{1}{2} \int_\Omega ||\nabla u||^2 d\tau$ is expressed as $\frac{1}{2} U^T S U$, the quantity $\frac{1}{2} \int_\Omega u^2 d\tau$ can be expressed as $\frac{1}{2} U^T T U$, where the T matrix is called the *metric* matrix. Like the Dirichlet matrix S, T can be constructed from individual element contributions.

$$\frac{1}{2} \int_{\Delta(\alpha\beta\gamma)} u^2 d\tau = \frac{1}{2} \sum_{j=\alpha,\beta,\gamma} \sum_{k=\alpha,\beta,\gamma} U_j T_{jk}^{(e)} U_k \tag{30}$$

where

$$T_{jk}^{(e)} = \int_{\Delta(\alpha\beta\gamma)} \psi_j(x, y) \psi_k(x, y) d\tau, \quad j, k \in \{\alpha, \beta, \gamma\} \tag{31}$$

First we evaluate $T_{jj}^{(e)} = \int_{\Delta(\alpha\beta\gamma)} \psi_j^2(x, y) d\tau$. We consider a differential strip parallel to the side opposite node j in the triangle $\alpha\beta\gamma$ If ψ_j changes by $d\psi_j$ on the strip, its width is $hd\psi_j$ and the length of the strip is $(1 - \psi_j)b$, where b is the length of the side opposite to node j and h is the height of the perpendicular from node j to the opposite side. So

$$T_{jj}^{(e)} = \int_{\Delta(\alpha\beta\gamma)} \psi_j^2 d\tau = \int_0^1 \psi_j^2 (1 - \psi_j) bh d\psi_j = \frac{bh}{12} = \frac{A}{6} \tag{32}$$

where A is the area of triangle $\alpha\beta\gamma$. Thus the diagonal terms of the element T matrix, $T_{\alpha\alpha}^{(e)}$, $T_{\beta\beta}^{(e)}$, and $T_{\gamma\gamma}^{(e)}$, are equal to $A/6$. What about the off-diagonal terms?

$$T_{\alpha\alpha}^{(e)} + T_{\alpha\beta}^{(e)} + T_{\alpha\gamma}^{(e)} = \int_{\Delta(\alpha\beta\gamma)} \psi_\alpha^2 + \psi_\alpha\psi_\beta + \psi_\alpha\psi_\gamma d\tau = \int_{\Delta(\alpha\beta\gamma)} \psi_\alpha(\psi_\alpha + \psi_\beta + \psi_\gamma) d\tau = \int_{\Delta(\alpha\beta\gamma)} \psi_\alpha d\tau \tag{33}$$

since $\psi_\alpha + \psi_\beta + \psi_\gamma = 1$. So

$$T_{\alpha\alpha}^{(e)} + T_{\alpha\beta}^{(e)} + T_{\alpha\gamma}^{(e)} = \int_{\Delta(\alpha\beta\gamma)} \psi_\alpha d\tau = \int_0^1 \psi_\alpha (1 - \psi_\alpha) bh d\psi_\alpha = \frac{bh}{6} = \frac{A}{3} \tag{34}$$

But $T_{\alpha\alpha}^{(e)} = A/6$. So

$$T_{\alpha\beta}^{(e)} + T_{\alpha\gamma}^{(e)} = \frac{A}{3} - \frac{A}{6} = \frac{A}{6} \tag{35}$$

Similarly we also have

$$T_{\beta\alpha}^{(e)} + T_{\beta\gamma}^{(e)} = \frac{A}{6} \tag{36}$$

$$T^{(e)}_{\gamma\alpha} + T^{(e)}_{\gamma\beta} = \frac{A}{6} \tag{37}$$

But $T^{(e)}_{\gamma\alpha} = T^{(e)}_{\alpha\gamma}$ etc. by the symmetry of the element T matrix. So adding the above three equations and dividing by 2 we see that

$$T^{(e)}_{\alpha\beta} + T^{(e)}_{\beta\gamma} + T^{(e)}_{\gamma\alpha} = \frac{A}{4} \tag{38}$$

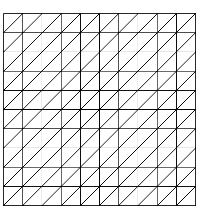
It it then seen that each off-diagonal element of the element T matrix is equal to $A/12$.

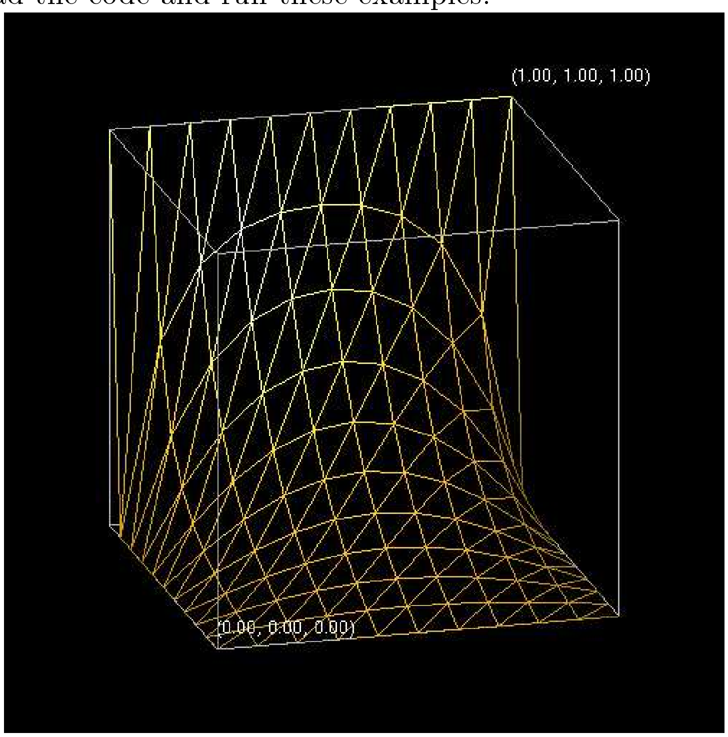
5 Example Code

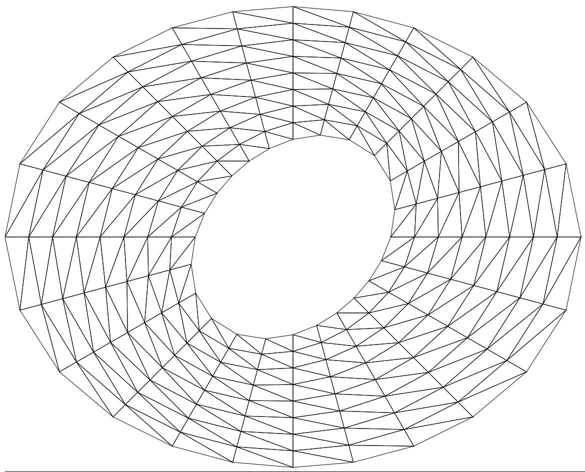
The example code available on the web site solves two problems.

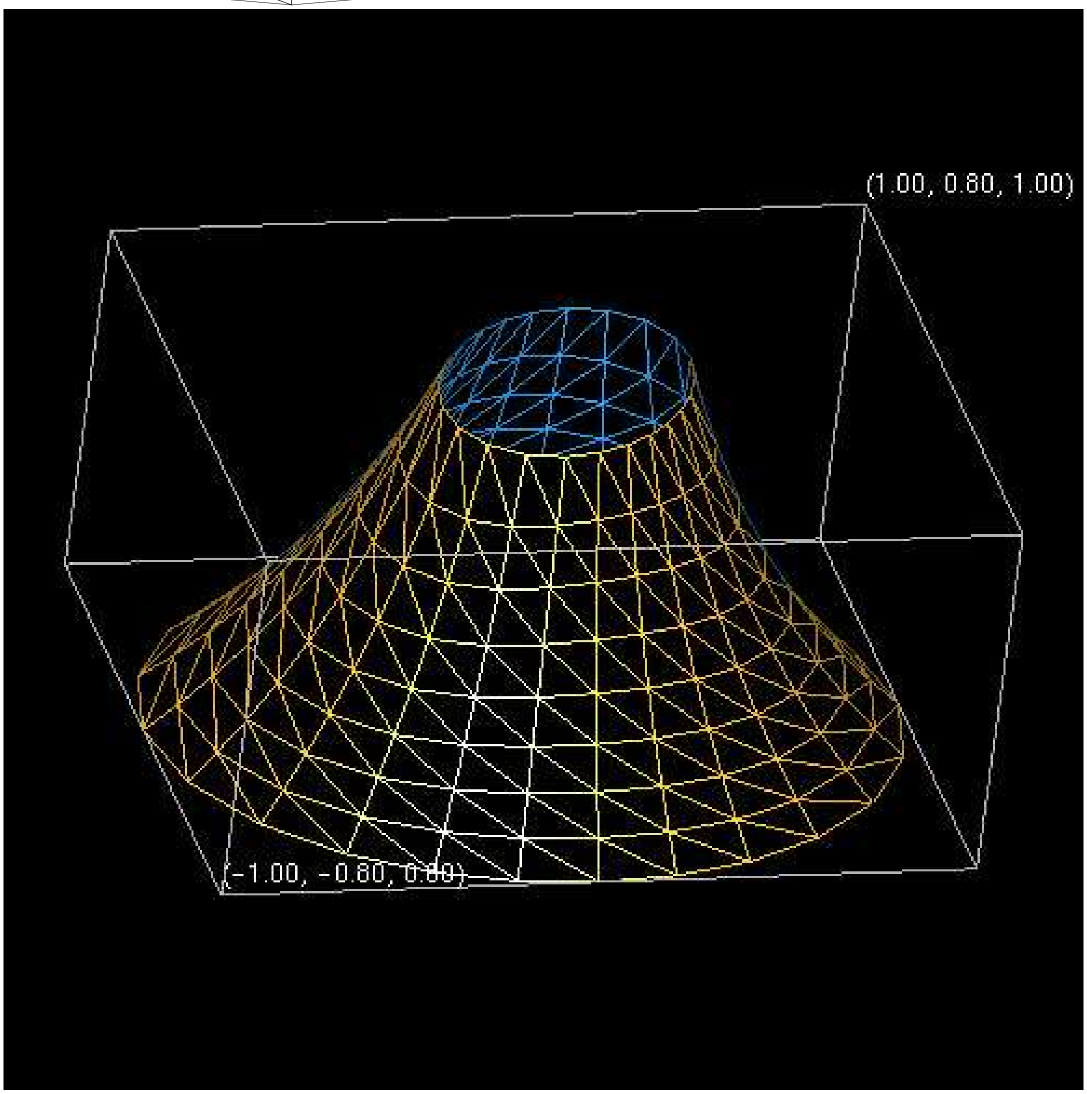
- Solution of Laplace equation on a rectangle: On three sides of the rectangle $u = 0$, while on the other side $u = 1$.
- Capacitance of a cable with elliptic cross sections for both the conductors.

Please download the code and run these examples.









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