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ALE-SUPG finite element method for convection–diffusion problems in time-dependent domains: Conservative form^{*}



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Sashikumaar Ganesan*, Shweta Srivastava

Department of Computational and Data Sciences, Indian Institute of Science, Bangalore 560012, India

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ABSTRACT

A Streamline Upwind Petrov–Galerkin (SUPG) finite element method for a convection dominated transient convection-diffusion-reaction equation in time-dependent domains is proposed. The time-dependent domain is handled by the arbitrary Lagrangian–Eulerian (ALE) approach, whereas the SUPG method is used for the spatial discretization. Further, the first order modified backward Euler and the second order modified Crank–Nicolson methods are used for the temporal discretization. It is shown that the stability of the semi-discrete (continuous in time) conservative ALE-SUPG equation is independent of the mesh velocity, whereas the stability of the fully discrete scheme with the implicit Euler time discretization is unconditionally stable and is only conditionally stable (time step depends on mesh velocity) with the Crank–Nicolson method. Numerical results are presented to support the stability estimates and to show the influence of the SUPG stabilization parameter in a time-dependent domain. Further, the proposed numerical scheme is applied to a boundary/layer problem in a time-dependent domain.

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1. Introduction

Numerical solution of a convection dominated transient convection-diffusion equation in time-dependent domains is considered in this paper. It is well-known that the standard Galerkin method induces spurious oscillations in the numerical solution of a convection dominated equation. In addition, the deformation of the domain makes the computations more complex. To suppress the oscillations in the solution and to enhance the stability of the numerical scheme, stabilization methods are used for convection dominated problems. The additional stabilization in stabilized numerical schemes mainly depends on the convection term in the equation. Moreover, the numerical method that we use to handle the domain movement modifies the convection term in the convection-diffusion equation.

In the Eulerian approach, a fixed mesh is used to solve the equations in time-dependent domains, and the moving boundaries/interfaces are captured using interface capturing/tracking methods such as Level-set [1], Volume of Fluid [2], Immersed boundary/Front-tracking method [3,4], etc. Therefore, the convection term is not modified when an Eulerian approach is used to handle time-dependent domains. Alternatively, the arbitrary Lagrangian–Eulerian (ALE) approach [5,6] is preferred when the application demands high accuracy and/or sharp moving boundaries/interfaces, for instance, in fluid-structure interaction applications. The moving boundaries/interfaces are resolved by a moving mesh and are tracked explicitly. Hence,

* Corresponding author.

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E-mail addresses: sashi@cds.iisc.ac.in, sashicds@gmail.com (S. Ganesan), shweta@cmg.cds.iisc.ac.in (S. Srivastava).

the ALE approach introduces a convective mesh velocity term into the model equation [7], and it alters the overall convective field of the problem. Since the mesh velocity need not be identical or even close to the convective velocity in many practical applications, the model problem can still be convection dominated and can have boundary/interior layers even after reformulating the model equations into an ALE form. It is of our interest in this paper.

Streamline Upwind Petrov–Galerkin (SUPG) is one of the popular stabilization methods for convection dominated problems, see for example [8–11] and references therein. Other popular stabilization methods such as Galerkin least-squares [12], edge stabilization [13], continuous interior penalty [14], local projection stabilization [15], orthogonal subgrid scale [16] have also been proposed in the literature for convection dominated problems in fixed domains, see [17] for an overview. A comparison of the SUPG method with other stabilization methods for a problem in fixed domain can be found in [18,19]. An adaptive SUPG method for a transient problem in fixed domain has been analyzed in [20].

For the problems in time-dependent domains, the analysis of standard Galerkin finite element discretization with different temporal discretizations has been presented in [6,7,21–24]. Further, the orthogonal subgrid scale method with the ALE approach for the solution of the transient convection–diffusion equation in a time-dependent domain has also been presented in [25]. However, to the best of the authors' knowledge, the SUPG discretization has not been studied for transient equations in time-dependent domains.

In this work, we analyze the SUPG finite element method for a convection dominated transient convection-diffusion equation in a time-dependent domain. In particular, an inconsistent SUPG discretization for a conservative ALE formulation of the transient equation is considered. The stability estimate for two temporal discretizations methods, (i) first order backward Euler and, (ii) second order Crank-Nicolson, with the inconsistent SUPG form are derived.

The paper is organized as follows. In Section 2, the transient convection-diffusion equation in a time-dependent domain and its ALE formulations are given. The spatial discretization using the SUPG finite element method is also presented in this section. Further, the stability of the semi-discrete problem (continuous in time) is derived in Section 3. Section 4 is devoted to the stability estimates of the fully discrete problem obtained with backward Euler and Crank-Nicolson time discretizations. Finally, the numerical studies are presented in Section 5.

2. Model problem and its ALE formulation

2.1. Convection-diffusion-reaction equation

Let T be a given time, and for $t \in [0, T]$ we consider a linear time-dependent convection-diffusion-reaction equation

where $\Omega_t \subset \mathbb{R}^d$, d = 2, 3 is a deforming/moving domain with the time-dependent boundary $\partial \Omega_t$. Here, u(t, x) is an unknown scalar function, ϵ is a constant diffusion coefficient, $\mathbf{b}(t, x)$ is a given convective velocity, c(t, x) is a reaction function, f(x) is a source term and $u_0(x)$ is a given initial data. We assume that Ω_t is bounded for each $t \in [0, T]$ with Lipschitz boundary, and the topology of domain Ω_t do not change in time. Further, assume that there exist a constant μ such that

$$0 < \mu \le \left(c(t, x) - \frac{1}{2}\nabla \cdot \mathbf{b}(t, x)\right), \quad \forall \ t \in (0, T), \ x \in \Omega_t.$$

$$\tag{2}$$

In addition, we assume that the given data are sufficiently smooth.

2.2. ALE formulation

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We now derive the arbitrary Lagrangian–Eulerian form of the considered model problem (1). Let $\hat{\Omega}$ be a reference domain, and define a family of bijective ALE mappings

 $\mathcal{A}_t: \hat{\Omega} \to \Omega_t, \quad \mathcal{A}_t(Y) = x(Y, t), \quad t \in (0, T).$

The initial domain Ω_0 or the previous time-step domain (when the deformation of the domain is large) can be considered as a reference domain $\hat{\Omega}$. Further, for a function $v \in C^0(\Omega_t)$ on the Eulerian frame, we define their corresponding function $\hat{v} \in C^0(\hat{\Omega})$ on the ALE frame by

$$\hat{\nu}: (0, \mathrm{T}) \times \hat{\Omega} \to \mathbb{R}, \quad \hat{\nu}:= \nu \circ \mathcal{A}_t, \quad \text{with} \quad \hat{\nu}(t, \mathrm{Y}) = \nu(t, \mathcal{A}_t(\mathrm{Y})).$$

Moreover, the time derivative on the ALE frame is defined as

$$\frac{\partial \nu}{\partial t}\Big|_{Y}: (0,T) \times \Omega_{t} \to \mathbb{R}, \quad \frac{\partial \nu}{\partial t}\Big|_{Y}(t,x) = \frac{\partial \hat{\nu}}{\partial t}(t,Y), \quad Y = \mathcal{A}_{t}^{-1}(x).$$

We now apply the chain rule to the time derivative of $v \circ A_t$ on the ALE frame to get

$$\frac{\partial \nu}{\partial t}\Big|_{Y} = \frac{\partial \nu}{\partial t}(t,x) + \frac{\partial x}{\partial t}\Big|_{Y} \cdot \nabla_{x}\nu = \frac{\partial \nu}{\partial t} + \frac{\partial \mathcal{A}_{t}(Y)}{\partial t} \cdot \nabla_{x}\nu = \frac{\partial \nu}{\partial t} + \mathbf{w} \cdot \nabla_{x}\nu,$$

where \mathbf{w} is the domain velocity. Using this relation in the model problem (1), we get the non-conservative ALE form

$$\frac{\partial u}{\partial t}\Big|_{Y} - \epsilon \Delta u + (\mathbf{b} - \mathbf{w}) \cdot \nabla u + cu = f.$$
(3)

Alternatively, we can use the Reynolds transport theorem

$$\frac{d}{dt} \int_{\Omega_t} u \, d\mathbf{x} = \int_{\hat{\Omega}} \frac{\partial (uJ_{A_t})}{\partial t} \, d\mathbf{Y} = \int_{\hat{\Omega}} \left[J_{A_t} \frac{\partial u}{\partial t} \Big|_{\mathbf{Y}} + u \frac{\partial J_{A_t}}{\partial t} \right] d\mathbf{Y} = \int_{\Omega_t} \left[\frac{\partial u}{\partial t} \Big|_{\mathbf{Y}} + u \nabla \cdot \mathbf{w} \right] d\mathbf{x}$$
$$= \int_{\Omega_t} \left(\frac{\partial u}{\partial t} + \mathbf{w} \cdot \nabla u + u \nabla \cdot \mathbf{w} \right) d\mathbf{x},$$

to derive the conservative ALE form of the equation. Here, the Euler expansion formula

$$\left. \frac{\partial \mathbf{J}_{A_t}}{\partial t} \right|_Y = \mathbf{J}_{A_t} \nabla \cdot \mathbf{w}$$
(4)

is used. Now, using the Reynolds transport theorem, the conservative ALE form of Eq. (1) can be written as

$$\left. \frac{\partial (uJ_{A_t})}{\partial t} \right|_{Y} + J_{A_t} \left[-\epsilon \Delta u + (\mathbf{b} - \mathbf{w}) \cdot \nabla u + (c - \nabla \cdot \mathbf{w}) u \right] = J_{A_t} f.$$
(5)

Here, $J_{A_t} = \det(\mathbf{J}_{A_t})$, and the Jacobian matrix of the ALE mapping, \mathbf{J}_{A_t} is given by

$$\mathbf{J}_{A_t} = \frac{\partial \mathbf{x}_i}{\partial Y_i}.$$

The main difference between (1) and (5) is the additional domain velocity **w** in the ALE form that accounts for the deformation of the domain. Moreover, the ALE mapping between two different time instances t_1 , $t_2 \in [0, T]$, that is the mapping, $A_{t_1,t_2} : \Omega_{t_1} \to \Omega_{t_2}$, is given by,

$$\mathcal{A}_{t_1,t_2} = \mathcal{A}_{t_2} \circ \mathcal{A}_{t_1}^{-1}.$$

2.3. Variational form of the conservative ALE equation

Let $H^1(\Omega_t)$ be the usual Sobolev space, and $H^1_0(\Omega_t)$ be a subspace of $H^1(\Omega_t)$ in which the functions vanish on the boundary $\partial \Omega_t$. Further, the solution space for Eq. (5) is defined as

$$V = \left\{ \nu \in H_0^1(\Omega_t), \quad \nu : (0,T] \times \Omega_t \to \mathbb{R}, \quad \nu = \hat{\nu} \circ A_t^{-1}, \quad \hat{\nu} \in H_0^1(\hat{\Omega}) \right\}$$

Moreover, the L^2 -inner product, the norm and the semi-norm, $(\cdot, \cdot)_t$, $\|\cdot\|_{0, t}$ and $\|\cdot\|_{1, t}$, respectively, over Ω_t are denoted by

$$(u, v)_t := \int_{\Omega_t} uv \, dx, \quad \|v\|_{0,t}^2 := (v, v)_t, \quad |v|_{1,t}^2 := (\nabla v, \nabla v)_t, \quad \forall \ u, v \in V.$$

Now, multiplying Eq. (5) with a test function $v \in V$ and integrate over Ω_t to get

$$\int_{\hat{\Omega}} \frac{\partial (uJ_{A_t})}{\partial t} \bigg|_{Y} \hat{v} \, dY + \int_{\Omega_t} \left[-\epsilon \, \Delta u + (\mathbf{b} - \mathbf{w}) \cdot \nabla u + (c - \nabla \cdot \mathbf{w}) u \right] v \, dx = \int_{\Omega_t} f v \, dx.$$

Using the fact that $\frac{\partial \hat{v}}{\partial t} = 0$, and after applying integration by parts to the higher order derivative term, the variational form of Eq. (5) reads:

For given $\hat{\Omega}$, **b**, **w**, *c*, u_0 and *f*, find $u \in V$ such that for all $t \in (0, T]$ and $v \in V$,

$$\frac{d}{dt}(u, v)_t + (\epsilon \nabla u, \nabla v)_t + ((\mathbf{b} - \mathbf{w}) \cdot \nabla u, v)_t + ((c - \nabla \cdot \mathbf{w})u, v)_t = (f, v)_t.$$
(6)

The stability analysis for the standard Galerkin finite element discretization (6) has been presented in [6,7,22]. Here, we analyze the SUPG discretization of the conservative ALE variational form (6).

2.4. SUPG discretization of the ALE equation

It is well-known that the standard Galerkin finite element discretization of convection–diffusion equation induces spurious oscillations in the numerical solution in convection dominated cases. Note that the convective term in the ALE variational form (6) is proportional to $(\mathbf{b} - \mathbf{w})$, and the instabilities and spurious oscillations are not expected when the mesh velocity is the same as convective velocity (pure Lagrangian form). However, this is not the case in the ALE form, and in practice, the mesh velocity need not be in the same direction as the convective velocity. Therefore, to circumvent the instabilities and to suppress the spurious oscillations, a stabilization method has to be used in practical applications, in particular,

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for problems with boundary and interior layers. One of the simplest and most popular stabilization method for convection dominated problems in fixed domains is the SUPG method, and it is considered here.

Let $\mathcal{T}_{h,t}$ be the collection of simplices obtained by triangulating the time-dependent domain Ω_t . We denote the diameter of the cell $K \in \mathcal{T}_{h,t}$ by $h_{K,t}$, and the global mesh size in the triangulated domain $\Omega_{h,t}$ by $h_t := \max\{h_{K,t} : K \in \mathcal{T}_{h,t}\}$. Suppose $V_h \subset V$ is a conforming finite element (finite dimensional) space. Let $\phi_h := \{\phi_i(x)\}, i = 1, 2, ..., \mathcal{N}$, be the finite element basis functions of V_h . The discrete finite element space V_h is then defined as

$$V_h = \left\{ u_h : u_h(t, x) = \sum_{i=1}^{\mathcal{N}} u_i(t)\phi_i(x); \quad u_i \in \mathbb{R} \right\} \subset H_0^1(\Omega_t).$$

$$\tag{7}$$

We next define the semi-discrete mesh velocity \mathbf{w}_h in space using the semi-discrete ALE mapping

$$\mathcal{A}_{h,t}:\hat{\Omega}_h \to \Omega_{h,t}.$$
(8)

Let

$$\mathcal{L}^{1}(\hat{\Omega}) = \left\{ \psi \in H^{1}(\hat{\Omega}) : \psi|_{K} \in P_{1}(\hat{K}) \text{ for all } \hat{K} \in \hat{\Omega}_{h} \right\},\$$

where P_1 is a set of polynomials of degree less than or equal to one on \hat{K} , be the piecewise linear Lagrangian finite element space. To move the mesh, it is sufficient to move the vertices, and thus the semi-discrete (continuous in time) mesh velocity $\hat{\mathbf{w}}_h(t, Y) \in \mathcal{L}^1(\hat{\Omega})^d$ in the ALE frame for each $t \in [0, T)$ is defined as

$$\hat{\mathbf{w}}_{h}(t,Y) = \sum_{i=1}^{\mathcal{M}} \mathbf{w}_{i}(t) \psi_{i}(Y); \quad \mathbf{w}_{i}(t) \in \mathbb{R}^{d}$$

Here, $\mathbf{w}_i(t)$ denotes the mesh velocity of the *i*th node of simplices at time *t* and $\psi_i(Y)$, i = 1, 2, ..., M, are the basis functions of $\mathcal{L}^1(\hat{\Omega})$. We then define the semi-discrete mesh velocity in the Eulerian frame as

$$\mathbf{w}_h(t, x) = \hat{\mathbf{w}}_h \circ \mathcal{A}_{h\,t}^{-1}(x).$$

Applying now the inconsistent SUPG finite element discretization to the ALE variational form (6), the semi-discrete form in space of (6) reads:

For given Ω_0 , $u_h(0, x) = u_0(x)$, **b**, \mathbf{w}_h , *c*, and *f*, find $u_h(t, x) \in V_h$ such that for all $t \in (0, T]$ and $v_h \in V_h$.

$$\frac{d}{dt}(u_h, v_h)_{h,t} + a_{SUPG}(u_h, v_h)_{h,t} - \int_{\Omega_{h,t}} \nabla \cdot (\mathbf{w}_h u_h) v_h dx = \int_{\Omega_{h,t}} f v_h dx + \sum_{K \in \mathcal{T}_{h,t}} \delta_K \int_K f (\mathbf{b} - \mathbf{w}_h) \cdot \nabla v_h dK, \tag{9}$$

where

a_{SUP}

$${}_{G}(u,v)_{h,t} = \epsilon \left(\nabla u, \nabla v\right)_{h,t} + (\mathbf{b} \cdot \nabla u, v)_{h,t} + (cu, v)_{h,t} + \sum_{K \in \mathcal{T}_{h,t}} \delta_{K}(-\epsilon \Delta u + (\mathbf{b} - \mathbf{w}_{h}) \cdot \nabla u + cu, (\mathbf{b} - \mathbf{w}_{h}) \cdot \nabla v)_{K}.$$

$$(10)$$

Here, $(\cdot, \cdot)_{h, t}$ and $(\cdot, \cdot)_K$ denotes the L^2 -inner product in $\Omega_{h, t}$ and in $K \in \mathcal{T}_{h, t}$, and δ_K is the SUPG (local) stabilization parameter. Further, $u_h(0, x) \in V_h$ is defined as the L^2 -projection of the initial value $u_0(x)$ onto V_h .

Lemma 1. Coercivity of $a_{SUPG}(\cdot, \cdot)_{h, t}$: Let the discrete form of the assumptions (2) be satisfied. Further, assume that the SUPG parameters satisfy

$$\delta_{K} \leq \frac{\mu_{0}}{2||c||_{K,\infty}^{2}}, \quad \delta_{K} \leq \frac{h_{K}^{2}}{2\epsilon c_{in\nu}^{2}}, \tag{11}$$

where c_{inv} is a constant used in inverse inequality. Then, the SUPG bilinear form satisfies

$$a_{SUPG}(u_h, u_h)_{h,t} \geq \frac{1}{2} |||u_h|||_t^2,$$

where the mesh-dependent norm is defined as

$$|||u|||_{t}^{2} = \left(\epsilon |u|_{1,t}^{2} + \sum_{K \in \mathcal{T}_{h,t}} \delta_{K}||(\mathbf{b} - \mathbf{w}_{h}) \cdot \nabla u||_{0,K}^{2} + \mu ||u||_{0,t}^{2}\right).$$

Proof. Using the assumption (2) in (10), we get

$$a_{SUPG}(u_h, u_h)_{h,t} \ge \epsilon |u_h|_{1,t}^2 + \mu ||u_h||_{0,t}^2 + \sum_{K \in \mathcal{T}_{h,t}} \delta_K ||(\mathbf{b} - \mathbf{w}_h) \cdot \nabla u_h||_{0,K}^2 + \sum_{K \in \mathcal{T}_{h,t}} \delta_K (-\epsilon \Delta u_h + c u_h, (\mathbf{b} - \mathbf{w}_h) \cdot \nabla u_h)_K.$$
(12)

Consider the last term in the above equation, apply Cauchy-Schwarz and then Young's inequalities to get

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$$\left| \sum_{K \in \mathcal{T}_{h,t}} \delta_{K}(-\epsilon \Delta u_{h} + cu_{h}, (\mathbf{b} - \mathbf{w}_{h}) \cdot \nabla u_{h})_{K} \right|$$

$$\leq \sum_{K \in \mathcal{T}_{h,t}} \left[\epsilon^{2} \delta_{K} ||\Delta u_{h}||_{0,K}^{2} + \delta_{K} c^{2} ||u_{h}||_{0,K}^{2} + \frac{1}{2} \delta_{K} ||(\mathbf{b} - \mathbf{w}_{h}) \cdot \nabla u_{h}||_{0,K}^{2} \right]$$

$$\leq \frac{1}{2} \left[\epsilon |u_{h}|_{1,K}^{2} + \mu ||u_{h}||_{0,K}^{2} + \delta_{K} ||(\mathbf{b} - \mathbf{w}_{h}) \cdot \nabla u_{h}||_{0,K}^{2} \right]$$

$$\leq \frac{1}{2} |||u_{h}|||_{t}^{2}.$$
(13)

Here, the inverse inequality

$$||\Delta u_h||_{0,K} = c_{in\nu}h_K^{-1}|u_h|_{1,K}, \ \forall \ u_h \in V_h$$

is used for the diffusive term. Note that the inverse inequality and the second assumption on δ_K in (11) can be omitted when piecewise linear finite elements are used. Finally, using the estimate (13) in (12), the coercivity is proved.

3. Stability of the semi-discrete (continuous in time) ALE-SUPG problem in space

3.1. Stability of the semi-discrete (continuous in time) conservative ALE-SUPG form

In the conservative semi-discrete form (9), we can not take $v_h = u_h$ to derive the stability estimate, since the variational form (6) is derived using $\frac{\partial v}{\partial t} = 0$. Nevertheless, a function $u_h \in V_h$ can be defined as a linear combination of basis functions of V_h with time-dependent coefficients, see (7), that is,

$$u_h(t,x) = \sum_{i \in \mathbb{N}} u_i(t)\phi_i(x)$$

Note that the basis functions ϕ_i do not depend on time in the reference domain, and thus

$$\left.\frac{\partial u_h}{\partial t}\right|_{Y}(t,x) = \sum_{i\in\mathbb{N}} \frac{du_i(t)}{dt}\phi_i(x).$$

Taking $v_h = \phi_i$ in (9) and multiplying the equation by $u_i(t) \in \mathbb{R}$, we get

$$\begin{aligned} u_i(t) \frac{d}{dt} \int_{\Omega_{h,t}} u_h \phi_i \, dx + \int_{\Omega_{h,t}} \epsilon \nabla u_h \nabla (u_i(t)\phi_i) \, dx + \int_{\Omega_{h,t}} \mathbf{b} \cdot \nabla u_h \, u_i(t)\phi_i \, dx + \int_{\Omega_{h,t}} c u_h \, u_i(t)\phi_i \, dx \\ &+ \sum_{K \in \mathcal{T}_{h,t}} \delta_K \Big(-\epsilon \,\Delta u_h + (\mathbf{b} - \mathbf{w}_h) \cdot \nabla u_h + c u_h, \, (\mathbf{b} - \mathbf{w}_h) \cdot \nabla u_i(t)\phi_i \Big)_K - \int_{\Omega_{h,t}} \nabla \cdot (\mathbf{w}_h \, u_h) \, u_i(t)\phi_i \, dx \\ &= \int_{\Omega_{h,t}} f \, u_i(t)\phi_i \, dx + \sum_{K \in \mathcal{T}_{h,t}} \delta_K(f, \, (\mathbf{b} - \mathbf{w}_h) \cdot \nabla (u_i(t)\phi_i))_K. \end{aligned}$$

Since $u_i(t)$ depends only on time, the first term can be rewritten as,

$$u_{i}(t)\frac{d}{dt}\int_{\Omega_{h,t}}u_{h}\phi_{i} dx = \frac{d}{dt}\int_{\Omega_{h,t}}u_{h}u_{i}(t)\phi_{i}dx - \int_{\Omega_{h,t}}u_{h}\phi_{i}\frac{du_{i}(t)}{dt} dx$$
$$= \frac{d}{dt}\int_{\Omega_{h,t}}u_{h}u_{i}(t)\phi_{i}dx - \int_{\Omega_{h,t}}u_{h}\frac{\partial\phi_{i}u_{i}(t)}{\partial t}\Big|_{Y} dx.$$

Now, summing over i = 1, 2, ..., N, we get

$$\begin{aligned} \frac{d}{dt} \|u_h\|_{0,t}^2 &- \int_{\Omega_{h,t}} u_h \frac{\partial u_h}{\partial t} \Big|_Y dx + \epsilon |u_h|_{1,t}^2 + \int_{\Omega_{h,t}} \mathbf{b} \cdot \nabla u_h u_h dx - \int_{\Omega_{h,t}} \nabla \cdot (\mathbf{w}_h u_h) u_h dx \\ &+ c ||u_h||_{0,t}^2 + \sum_{K \in \mathcal{T}_{h,t}} \delta_K \Big(-\epsilon \Delta u_h + (\mathbf{b} - \mathbf{w}_h) \cdot \nabla u_h + c u_h, (\mathbf{b} - \mathbf{w}_h) \cdot \nabla u_h \Big)_K \\ &= \int_{\Omega_{h,t}} f u_h dx + \sum_{K \in \mathcal{T}_{h,t}} \delta_K (f, (\mathbf{b} - \mathbf{w}_h) \cdot \nabla u_h)_K. \end{aligned}$$

Using the Euler expansion (4), the second term can be written as

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$$\begin{split} \int_{\Omega_{h,t}} \frac{\partial u_h}{\partial t} \bigg|_Y u_h \, dx &= \int_{\widehat{\Omega}} \frac{\partial u_h}{\partial t} \, u_h \, J_{A_t} \, dY = \frac{1}{2} \frac{d}{dt} \int_{\widehat{\Omega}} (u_h^2 \, J_{A_t}) \, dY - \frac{1}{2} \int_{\widehat{\Omega}} u_h^2 \frac{\partial J_{A_t}}{\partial t} \, dY \\ &= \frac{1}{2} \frac{d}{dt} \int_{\widehat{\Omega}} u_h^2 \, J_{A_t} \, dY - \frac{1}{2} \int_{\widehat{\Omega}} u_h^2 \nabla \cdot \mathbf{w}_h \, J_{A_t} \, dY \\ &= \frac{1}{2} \left(\frac{d}{dt} ||u_h||_{0,t}^2 - \int_{\Omega_{h,t}} u_h^2 \nabla \cdot \mathbf{w}_h dx \right). \end{split}$$

Further, applying the Cauchy-Schwarz and Young's inequalities, the right hand side terms become

$$|(f, u_h)| = \left| \left(\frac{f}{\mu^{1/2}}, \mu^{1/2} u_h \right) \right| \le \frac{1}{\mu} ||f||_{0,t}^2 + \frac{1}{4} \mu ||u_h||_{0,t}^2$$

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$$\left|\sum_{K\in\mathcal{T}_{h,t}}\delta_K(f,(\mathbf{b}-\mathbf{w}_h)\cdot\nabla u_h)_K\right|\leq \sum_{K\in\mathcal{T}_{h,t}}\delta_K||f||_{0,K}^2+\frac{1}{4}\sum_{K\in\mathcal{T}_{h,t}}\delta_K||(\mathbf{b}-\mathbf{w}_h)\cdot\nabla u_h||_{0,K}^2.$$

Using the above relations, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}||u_{h}||_{0,t}^{2} + \frac{1}{2}|||u_{h}|||_{t}^{2} + \frac{1}{2}\int_{\Omega_{h,t}}u_{h}^{2}\nabla\cdot\mathbf{w}_{h}dx - \int_{\Omega_{h,t}}\nabla\cdot(\mathbf{w}_{h}\ u_{h})\ u_{h}\ dx \\ &\leq \frac{1}{\mu}||f||_{0,t}^{2} + \frac{1}{4}||\mu^{1/2}u_{h}||_{0,t}^{2} + \sum_{K\in\mathcal{T}_{h,t}}\delta_{K}||f||_{0,K}^{2} + \frac{1}{4}\sum_{K\in\mathcal{T}_{h,t}}\delta_{K}||(\mathbf{b}-\mathbf{w}_{h})\cdot\nabla u_{h})||_{0,K}^{2} \\ &\leq \frac{1}{\mu}||f||_{0,t}^{2} + \sum_{K\in\mathcal{T}_{h,t}}\delta_{K}||f||_{0,K}^{2} + \frac{1}{4}|||u_{h}|||_{t}^{2}. \end{aligned}$$

Further, applying integration by parts to the last term on left hand side, that is

$$\int_{\Omega_{h,t}} \nabla \cdot (\mathbf{w}_h \ u_h) \ u_h \ dx = -\int_{\Omega_{h,t}} \mathbf{w}_h \cdot \nabla u_h \ u_h \ dx = \frac{1}{2} \int_{\Omega_{h,t}} u_h^2 \nabla \cdot \mathbf{w}_h dx,$$

we get

$$\frac{d}{dt}||u_h||_{0,t}^2+\frac{1}{2}|||u_h|||_t^2\leq \frac{2}{\mu}||f||_{0,t}^2+2\sum_{K\in\mathcal{T}_{h,t}}\delta_K||f||_{0,K}^2.$$

Finally, integrating the above inequality over time (0, T), we get the stability estimate

$$||u_{h}||_{0,t}^{2} + \frac{1}{2} \int_{0}^{T} |||u_{h}|||_{t}^{2} dt \leq ||u_{h}(0)||_{0,t}^{2} + \frac{2}{\mu} \int_{0}^{T} ||f||_{0,t}^{2} dt + 2 \int_{0}^{T} \sum_{K \in \mathcal{T}_{h,t}} \delta_{K} ||f||_{0,K}^{2} dt$$

which is independent of the mesh velocity \mathbf{w}_h .

4. Fully discrete scheme

In this section, we present the stability estimates for a fully discrete conservative ALE-SUPG form. In particular, the first order modified implicit backward Euler and the second order modified Crank–Nicolson time discretizations are analyzed.

4.1. Discrete ALE-SUPG with Implicit Euler method

Let $0 = t^0 < t^1 < \cdots < t^N = T$ be a decomposition of the considered time interval [0, T] into N equal time intervals. Let us denote the uniform time step by $\Delta t = t^n - t^{n-1}$, $1 \le n \le N$. Further, let u_h^n be an approximation of $u(t^n, x)$ in $V_h \subset H_0^1(\Omega_{t^n})$, where Ω_{t^n} is the deforming domain at time $t = t^n$. We first discretize the ALE mapping in time using a linear interpolation. We denote the discrete ALE mapping by $A_{h,\Delta t}$, and define it for every $\tau \in [t^n, t^{n+1}]$ by

$$\mathcal{A}_{h,\Delta t}(\mathbf{Y}) = \frac{\tau - t^n}{\Delta t} \mathcal{A}_{h,t^{n+1}}(\mathbf{Y}) + \frac{t^{n+1} - \tau}{\Delta t} \mathcal{A}_{h,t^n}(\mathbf{Y}),$$

where $A_{h,t}(Y)$ is the time continuous ALE mapping defined in (8). Since the discrete ALE mapping is defined linearly in time, we obtain the discrete mesh velocity

$$\hat{\mathbf{w}}_{h}^{n+1}(Y) = \frac{\mathcal{A}_{h,t^{n+1}}(Y) - \mathcal{A}_{h,t^{n}}(Y)}{\Delta t}$$

as a piecewise constant function in time. Further, we define the mesh velocity on the Eulerian frame as

$$\mathbf{W}_{h}^{n+1} = \hat{\mathbf{W}}_{h}^{n+1} \circ \mathcal{A}_{h \wedge t}^{-1}(x).$$

In the derivation of stability estimates, we use the geometric conservative law (GCL)

$$\int_{\Omega_{t^{n+1}}} \phi_i \phi_j \, dx - \int_{\Omega_{t^n}} \phi_i \phi_j \, dx = \int_{t^n}^{t^{n+1}} \int_{\Omega_{\tau}} \phi_i(x) \phi_j(x) \, \nabla \cdot \mathbf{w}_h(\tau) \, dx \, d\tau,$$

see [7] for more details. Since the mesh velocity is assumed to be a piecewise constant function in time, the midpoint time integration rule can be applied to the GCL to get

$$\int_{\Omega_{t^{n+1}}} \phi_i \phi_j \, dx - \int_{\Omega_{t^n}} \phi_i \phi_j \, dx = \Delta t \int_{\Omega_{t^{n+1/2}}} \phi_i(x) \, \psi_j(x) \, \nabla \cdot \mathbf{w}_h \, dx. \tag{14}$$

Further, to avoid complex notations, we do not explicitly write the ALE mappings inside the integrals, that is, the inner product $(u_h^m, v_h^n)_{h,m}$, for $m \neq n$ denotes

$$\int_{\Omega_{t^m}} u_h^m v_h^n \, dx = \int_{\Omega_{t^m}} u_h^m v_h^n \circ \mathcal{A}_{t^n, t^m}(x) \, dx$$

integrals to be evaluated in the backward Euler method, since it is necessary for the

Remark 1. Note that in order to satisfy the GCL condition (14), an integral with $\nabla \cdot \mathbf{w}$ term that arise from (9) needs to be evaluated exactly in time at $t = t^{n+1/2}$.

Applying the midpoint time integration rule to the semi-discrete problem (9), the ALE-SUPG backward Euler discrete form of (9) reads: For given Ω_0 , $u_k(0) = u_{k,0}$, **b**, $\mathbf{w}^{n+1/2}$, c_k and $f^{n+1/2}$ find $u^{n+1} \in V_k$ in the time interval (t^n, t^{n+1}) such that for all $v_k \in V_k$

$$\frac{1}{\Delta t} \Big[(u_h^{n+1}, v_h)_{\Omega_{h,t^{n+1}}} - (u_h^n, v_h)_{\Omega_{h,t^n}} \Big] + a_{SUPG}^{n+1/2} (u_h^{n+1}, v_h) - \int_{\Omega_{h,t^{n+1/2}}} \nabla \cdot (\mathbf{w}_h u_h^{n+1}) v_h \, dx$$

$$= \int_{\Omega_{h,t^{n+1/2}}} f^{n+1/2} v_h \, dx + \left(\sum_{K \in \mathcal{T}_{h,t^{n+1/2}}} \delta_K \int_K f^{n+1/2} \, (\mathbf{b} - \mathbf{w}_h) \cdot \nabla v_h \, dK \right),$$
(15)

where

$$a_{SUPG}^{n+1/2}(u_h, v_h) = \epsilon \left(\nabla u_h, \nabla v_h\right)_{\Omega_{h,t^{n+1/2}}} + (\mathbf{b} \cdot \nabla u_h, v_h)_{\Omega_{h,t^{n+1/2}}} + (cu_h, v_h)_{\Omega_{h,t^{n+1/2}}} + \sum_{K \in \mathcal{T}_{h,t^{n+1/2}}} \delta_K(-\epsilon \Delta u_h + (\mathbf{b} - \mathbf{w}_h) \cdot \nabla u_h + cu_h, \ (\mathbf{b} - \mathbf{w}_h) \cdot \nabla v_h)_K.$$

Lemma 2 (Stability estimate for the conservative ALE-SUPG form with modified implicit Euler method). Let the discrete version of (2) and the assumption (11) on δ_K hold true. Further, assume that $\delta_K \leq \frac{\Delta t}{4}$, the solution of the conservative problem (15) satisfies

$$||u_{h}^{n+1}||_{0,t^{N+1}}^{2} + \frac{\Delta t}{2} \sum_{n=0}^{N} |||u_{h}^{n+1}|||_{t^{n+1/2}}^{2} \leq ||u_{h}^{0}||_{0,t^{0}}^{2} + \frac{2\Delta t}{\mu} \sum_{n=0}^{N} ||f^{n+1/2}||_{0,t^{n+1/2}}^{2} + 2\Delta t \sum_{K \in \mathcal{T}_{h,t^{n+1/2}}} \delta_{K} \sum_{n=0}^{N} ||f^{n+1/2}||_{0,K^{n+1/2}}^{2} + 2\Delta t \sum_{K \in \mathcal{T}_{h,t^{n+1/2}}} ||f^{n+1/2}||_{0,K^{n+1/2}}^{2} +$$

Proof. Rewrite Eq. (15) to get

$$\frac{1}{\Delta t} \int_{\Omega_{h,t^{n+1}}} u_h^{n+1} v_h \, dx - \frac{1}{\Delta t} \int_{\Omega_{h,t^n}} u_h^n v_h \, dx + \int_{\Omega_{h,t^{n+1/2}}} \mathbf{b} \cdot \nabla u_h^{n+1} \, v_h \, dx \\
- \int_{\Omega_{h,t^{n+1/2}}} \nabla \cdot (\mathbf{w}_h u_h^{n+1}) \, v_h \, dx + \int_{\Omega_{h,t^{n+1/2}}} \epsilon \nabla u_h^{n+1} \nabla v_h \, dx + \int_{\Omega_{h,t^{n+1/2}}} c u_h^{n+1} v_h \, dx \\
+ \sum_{K \in \mathcal{T}_{h,t^{n+1/2}}} \delta_K \Big(-\epsilon \Delta u_h^{n+1} + ((\mathbf{b} - \mathbf{w}_h) \cdot \nabla u_h^{n+1}) + c u_h^{n+1}, \, (\mathbf{b} - \mathbf{w}_h) \cdot \nabla v_h \Big)_K \\
= \int_{\Omega_{h,t^{n+1/2}}} f^{n+1/2} \, v_h \, dx + \sum_{K \in \mathcal{T}_{h,t^{n+1/2}}} \delta_K \int_K f^{n+1/2} \, (\mathbf{b} - \mathbf{w}_h) \cdot \nabla v_h \, dK.$$

Taking $v_h = u_h^{n+1}$ and applying integration by parts for the mesh velocity term, we get

$$\int_{\Omega_{h,t^{n+1/2}}} u_h^{n+1} \nabla \cdot (\mathbf{w}_h u_h^{n+1}) \ dx = -\int_{\Omega_{h,t^{n+1/2}}} \nabla u_h^{n+1} \cdot (\mathbf{w}_h u_h^{n+1}) \ dx = \frac{1}{2} \int_{\Omega_{h,t^{n+1/2}}} \nabla \cdot \mathbf{w}_h (u_h^{n+1})^2 dx.$$

Using this relation in the above equation, we get,

$$\int_{\Omega_{h,t^{n+1}}} u_h^{n+1} u_h^{n+1} dx - \int_{\Omega_{h,t^n}} u_h^n u_h^{n+1} dx + \Delta t \ a_{SUPG}(u_h^{n+1}, u_h^{n+1})_{h,t^{n+1/2}} - \frac{\Delta t}{2} \int_{\Omega_{h,t^{n+1/2}}} \nabla \cdot \mathbf{w}_h |u_h^{n+1}|^2 dx$$

$$\leq \Delta t \int_{\Omega_{h,t^{n+1/2}}} f^{n+1/2} \ u_h^{n+1} \ dx + \sum_{K \in \mathcal{T}_{h,t^{n+1/2}}} \delta_K \Delta t \int_K f^{n+1/2} \ (\mathbf{b} - \mathbf{w}_h) \cdot \nabla u_h^{n+1} \ dK.$$

Applying the coercivity of the bilinear form and the Cauchy-Schwarz inequality, we get

$$\begin{split} |u_{h}^{n+1}||_{0,t^{n+1}}^{2} &+ \frac{\Delta t}{2} |||u_{h}^{n+1}|||_{t^{n+1/2}}^{2} - \frac{\Delta t}{2} \int_{\Omega_{h,t^{n+1/2}}} \nabla \cdot \mathbf{w}_{h} |u_{h}^{n+1}|^{2} dx \\ &\leq \int_{\Omega_{h,t^{n}}} u_{h}^{n} u_{h}^{n+1} dx + \Delta t \int_{\Omega_{h,t^{n+1/2}}} f^{n+1/2} u_{h}^{n+1} dx + \sum_{K \in \mathcal{T}_{h,t^{n+1/2}}} \delta_{K} \Delta t \int_{K} f^{n+1/2} (\mathbf{b} - \mathbf{w}_{h}) \cdot \nabla u_{h}^{n+1} dK \\ &\leq \frac{1}{2} (||u_{h}^{n+1}||_{0,t^{n}}^{2} + ||u_{h}^{n}||_{0,t^{n}}^{2}) + \frac{\Delta t}{\mu} ||f^{n+1/2}||_{0,t^{n+1/2}}^{2} + \Delta t \sum_{K \in \mathcal{T}_{h,t^{n+1/2}}} \delta_{K} ||f^{n+1/2}||_{0,K}^{2} + \frac{\Delta t}{4} |||u_{h}^{n+1}|||_{t^{n+1/2}}^{2}. \end{split}$$

Further, using the relation,

$$||u_h^{n+1}||_{0,t^{n+1}}^2 - ||u_h^{n+1}||_{0,t^n}^2 = \int_{t^n}^{t^{n+1}} \int_{\Omega_{h,t}} |u_h^{n+1}|^2 \nabla \cdot \mathbf{w}_h \ dx = \Delta t \int_{\Omega_{h,t^{n+1/2}}} |u_h^{n+1}|^2 \nabla \cdot \mathbf{w}_h \ dx,$$

which can be derived from GCL equation (14) with $\psi_i = \psi_j = u_h^{n+1}$, we get,

$$||u_{h}^{n+1}||_{0,t^{n+1}}^{2} + \frac{\Delta t}{2}|||u_{h}^{n+1}|||_{t^{n+1/2}}^{2} \leq ||u_{h}^{n}||_{0,t^{n}}^{2} + 2\Delta t \sum_{K \in \mathcal{T}_{h,t^{n+1/2}}} \delta_{K}||f^{n+1/2}||_{0,K}^{2} + \frac{2\Delta t}{\mu}||f^{n+1/2}||_{0,t^{n+1/2}}^{2}.$$

Finally, summing over all time steps, we get the unconditional stability estimate. \Box

4.2. Discrete ALE-SUPG with Crank-Nicolson method

We next consider the modified Crank-Nicolson method which is basically the Runge-Kutta method of order 2. For an equation

$$\frac{du(t)}{dt} = f(u(t), t), \quad t > 0 \text{ and } u(0) = u_0,$$

the modified Crank-Nicolson scheme reads

$$u^{n+1} - u^n = \Delta t f\left(\frac{u^{n+1} + u^n}{2}, t^{n+\frac{1}{2}}\right).$$

Applying the modified Crank–Nicolson scheme to (9), the fully discrete form of (9) reads: For given Ω_0 , $u_h(0) = u_{h,0}$, **b**, $\mathbf{w}_h^{n+1/2}$, c, and $f^{n+1/2}$ find $u_h^{n+1} \in V_h$ in the time interval (t^n, t^{n+1}) such that for all $v_h \in V_h$

$$\frac{1}{\Delta t} \Big[(u_h^{n+1}, v_h)_{\Omega_{h,t^{n+1}}} - (u_h^n, v_h)_{\Omega_{h,t^n}} \Big] - \left(\nabla \cdot \left(\mathbf{w}_h^{n+1/2} \left(\frac{u_h^{n+1} + u_h^n}{2} \right) \right), v_h \right)_{\Omega_{h,t^{n+1/2}}} + a_{SUPG}^{n+1/2} \left(\frac{u_h^{n+1} + u_h^n}{2}, v_h \right) = \left(f^{n+1/2} v_h \right)_{\Omega_{h,t^{n+1/2}}} + \sum_{K \in \mathcal{T}_{h,t^{n+1/2}}} \delta_K \left(f^{n+1/2} \left(\mathbf{b} - \mathbf{w}_h \right) \cdot \nabla v_h \right)_K.$$
(16)

Let us recall the following discrete Gronwall inequality, which will be used in the stability estimate of the conservative ALE-SUPG form with Crank-Nicolson method.

Lemma 3 (Gronwall lemma, [26]). Let $f_i A_n, B_n, C_n$ be given sequences of non-negative numbers for $n \ge 0$ such that the following inequality holds

$$A_n + \Delta t \sum_{i=0}^n B_i \leq \Delta t \sum_{i=0}^n \gamma_i A_i + \Delta t \sum_{i=0}^n C_i + f.$$

Suppose that $\Delta t \gamma_i < 1$ for all i, and set $\sigma_i = (1 - \Delta t \gamma_i)^{-1}$. We then have

$$A_n + \Delta t \sum_{i=0}^n B_i \leq \exp\left(\Delta t \sum_{i=0}^n \sigma_i \gamma_i\right) \left[\Delta t \sum_{i=0}^n C_i + f\right].$$

Lemma 4 (Stability estimate for the conservative ALE-SUPG form with Crank–Nicolson method). Let the discrete version of (2) and the assumption (11) on δ_K hold true. Further, assume that

$$\Delta t < \frac{1}{\beta_1^n + \beta_2^n} = \left(||\nabla \cdot \mathbf{w}_h||_{\infty, t^{n+1/2}} ||\mathcal{J}_{\mathcal{A}_{t^{n+1/2}, t^{n+1}}}||_{\infty, t^{n+1}} + ||\nabla \cdot \mathbf{w}_h||_{\infty, t^{n+1/2}} ||\mathcal{J}_{\mathcal{A}_{t^n, t^{n+1/2}}}||_{\infty, t^n} \right)^{-1}$$

and $\delta_K \leq \frac{\Delta t}{4}$, then

$$\begin{split} \|u_{h}^{N+1}\|_{0,t^{N+1}}^{2} &+ \frac{\Delta t}{4} \sum_{n=0}^{N} |||u_{h}^{n+1} + u_{h}^{n}|||_{t^{n+1/2}}^{2} \\ &\leq \left((1 + \Delta t \beta_{2}^{0}) \|u_{h}^{0}\|_{0,t^{0}}^{2} + \Delta t \sum_{n=0}^{N} \left(\frac{2}{\mu} + \Delta t\right) \|f^{n+1/2}\|_{0,t^{n+1/2}}^{2} \right) \exp\left(\Delta t \sum_{n=1}^{N+1} \frac{\beta_{1}^{n} + \beta_{2}^{n}}{1 - \Delta t (\beta_{1}^{n} + \beta_{2}^{n})} \right). \end{split}$$

Here, $|| \cdot ||_{\infty,t^n}$ is the L^{∞} -norm in Ω_{t^n} .

Proof. Testing Eq. (16) with $v_h = u_h^{n+1} + u_h^n$, and using the relation

$$(u_h, u_h + v_h) = \frac{1}{2} ||u_h||^2 + \frac{1}{2} ||u_h + v_h||^2 - \frac{1}{2} ||v_h||^2$$

and then the GCL (14) condition, the first term can be written as,

$$\begin{split} &\int_{\Omega_{h,t^{n+1}}} u_h^{n+1}(u_h^{n+1}+u_h^n) \, dx \, - \int_{\Omega_{h,t^n}} u_h^n(u_h^{n+1}+u_h^n) \, dx \\ &= \frac{1}{2} ||u_h^{n+1}||_{0,t^{n+1}}^2 + \frac{1}{2} ||u_h^{n+1}+u_h^n||_{0,t^{n+1}}^2 - \frac{1}{2} ||u_h^n||_{0,t^n}^2 - \frac{1}{2} ||u_h^{n+1}+u_h^n||_{0,t^n}^2 + \frac{1}{2} ||u_h^{n+1}||_{0,t^n}^2 \\ &= \frac{1}{2} ||u_h^{n+1}||_{0,t^{n+1}}^2 - \frac{1}{2} ||u_h^n||_{0,t^n}^2 + \frac{1}{2} (||u_h^{n+1}+u_h^n||_{0,t^{n+1}}^2 - ||u_h^{n+1}+u_h^n||_{0,t^n}^2) - \frac{1}{2} ||u_h^n||_{0,t^{n+1}}^2 + \frac{1}{2} ||u_h^{n+1}||_{0,t^n}^2 \\ &= \frac{1}{2} ||u_h^{n+1}||_{0,t^{n+1}}^2 - \frac{1}{2} ||u_h^n||_{0,t^n}^2 + \frac{\Delta t}{2} \left(\int_{\Omega_{t^{n+1/2}}} \nabla \cdot \mathbf{w}_h |(u_h^{n+1}+u_h^n)|^2 \, dx \right) \\ &- \frac{1}{2} \left(||u_h^n||_{0,t^n}^2 + \Delta t \int_{\Omega_{t^{n+1/2}}} \nabla \cdot \mathbf{w}_h |u_h^{n+2} \, dx \right) + \frac{1}{2} \left(||u_h^{n+1}||_{0,t^{n+1}}^2 - \Delta t \int_{\Omega_{t^{n+1/2}}} \nabla \cdot \mathbf{w}_h |u_h^{n+1}|^2 \, dx \right) \\ &= ||u_h^{n+1}||_{0,t^{n+1}}^2 - ||u_h^n||_{0,t^n}^2 + \Delta t \int_{\Omega_{t^{n+1/2}}} \nabla \cdot \mathbf{w}_h u_h^{n+1} u_h^n \, dx. \end{split}$$

Applying integration by parts to the mesh velocity term, along with the coercivity of the bilinear form and using the Cauchy–Schwarz inequality to the right hand side terms, we get

$$\begin{split} ||u_{h}^{n+1}||_{0,t^{n+1}}^{2} + \frac{\Delta t}{8} |||(u_{h}^{n+1} + u_{h}^{n})|||_{t^{n+1/2}}^{2} + \Delta t \int_{\Omega_{h,t^{n+1/2}}} \nabla \cdot \mathbf{w}_{h} u_{h}^{n+1} u_{h}^{n} dx \\ \leq ||u_{h}^{n}||_{0,t^{n}}^{2} + \frac{\Delta t}{4} \int_{\Omega_{h,t^{n+1/2}}} \nabla \cdot \mathbf{w}_{h} |u_{h}^{n+1} + u_{h}^{n}|^{2} dx + \frac{\Delta t}{\mu} ||f^{n+1/2}||_{0,t^{n+1/2}}^{2} + \Delta t \sum_{K \in \mathcal{T}_{h,t^{n+1/2}}} \delta_{K} ||f^{n+1/2}||_{K}^{2} \\ \leq \Delta t \int_{\Omega_{h,t^{n+1/2}}} \nabla \cdot \mathbf{w}_{h} \Big(\frac{1}{4} |u_{h}^{n+1} + u_{h}^{n}|^{2} - u_{h}^{n+1} u_{h}^{n} \Big) dx + ||u_{h}^{n}||_{0,t^{n}}^{2} + \frac{\Delta t}{\mu} ||f^{n+1/2}||_{0,t^{n+1/2}}^{2} + \Delta t \sum_{K \in \mathcal{T}_{h,t^{n+1/2}}} \delta_{K} ||f^{n+1/2}||_{K}^{2} \\ \leq \frac{\Delta t}{2} \int_{\Omega_{h,t^{n+1/2}}} \nabla \cdot \mathbf{w}_{h} \Big(|u_{h}^{n}|^{2} + |u_{h}^{n+1}|^{2} \Big) dx + ||u_{h}^{n}||_{0,t^{n}}^{2} + \frac{\Delta t}{\mu} ||f^{n+1/2}||_{0,t^{n+1/2}}^{2} + \Delta t \sum_{K \in \mathcal{T}_{h,t^{n+1/2}}} \delta_{K} ||f^{n+1/2}||_{K}^{2}. \end{split}$$

Using the ALE mapping and its Jacobian, we obtain

$$\begin{split} ||u_{h}^{n+1}||_{0,t^{n+1}}^{2} + \frac{\Delta t}{8} |||u_{h}^{n+1} + u_{h}^{n}|||_{t^{n+1/2}}^{2} \leq \frac{\Delta t}{2} ||\nabla \cdot \mathbf{w}_{h}||_{\infty,t^{n+1/2}} ||\mathcal{J}_{\mathcal{A}_{t^{n+1}, t^{n+1/2}}}||_{\infty,t^{n+1}} ||u_{h}^{n+1}||_{0,t^{n+1}}^{2} + \frac{\Delta t}{\mu} ||f^{n+1/2}||_{0,t^{n+1/2}}^{2} \\ + \Delta t \sum_{K \in \mathcal{T}_{h,t^{n+1/2}}} \delta_{K} ||f^{n+1/2}||_{K}^{2} + \frac{\Delta t}{2} ||\nabla \cdot \mathbf{w}_{h}||_{\infty,t^{n+1/2}} ||\mathcal{J}_{\mathcal{A}_{t^{n}, t^{n+1/2}}}||_{\infty,t^{n}} ||u_{h}^{n}||_{0,t^{n}}^{2}. \end{split}$$

Denoting

$$\beta_1^n = \frac{1}{2} ||\nabla \cdot \mathbf{w}_h||_{\infty, t^{n+1/2}} ||\mathcal{J}_{\mathcal{A}_{t^{n+1}, t^{n+1/2}}}||_{\infty, t^{n+1}}, \quad \beta_2^n = \frac{1}{2} ||\nabla \cdot \mathbf{w}_h||_{\infty, t^{n+1/2}} ||\mathcal{J}_{\mathcal{A}_{t^n, t^{n+1/2}}}||_{\infty, t^n}$$

the above inequality becomes

$$\begin{split} ||u_{h}^{n+1}||_{0,t^{n+1}}^{2} + \frac{\Delta t}{8} |||(u_{h}^{n+1} + u_{h}^{n})|||_{t^{n+1/2}}^{2} &\leq \Delta t \beta_{1}^{n+1} ||u_{h}^{n+1}||_{0,t^{n+1}}^{2} + (1 + \Delta t \beta_{2}^{n})||u_{h}^{n}||_{0,t^{n}}^{2} \\ &+ \frac{\Delta t}{\mu} ||f^{n+1/2}||_{0,t^{n+1/2}}^{2} + \Delta t \sum_{K \in \mathcal{T}_{h,t^{n+1/2}}} \delta_{K} ||f^{n+1/2}||_{K}^{2}. \end{split}$$

Summing over the index n = 0, 1, 2, ... N, and using the assumption on δ_K , we get

$$\begin{split} ||u_{h}^{N+1}||_{0,t^{N+1}}^{2} + \frac{\Delta t}{8} \sum_{n=0}^{N} |||(u_{h}^{n+1} + u_{h}^{n})|||_{t^{n+1/2}}^{2} \leq \Delta t \beta_{1}^{N+1} ||u_{h}^{N+1}||_{0,t^{N+1}}^{2} + \Delta t \sum_{n=1}^{N} (\beta_{1}^{n} + \beta_{2}^{n})||u_{h}^{n}||_{0,t^{n}}^{2} + (1 + \Delta t \beta_{2}^{0})||u_{h}^{0}||_{0,t^{0}}^{2} \\ + \frac{\Delta t}{\mu} ||f^{n+1/2}||_{0,t^{n+1/2}}^{2} + \Delta t \sum_{K \in \mathcal{T}_{h,t^{n+1/2}}} \delta_{K} ||f^{n+1/2}||_{K}^{2} \\ \leq \Delta t \sum_{n=1}^{N+1} (\beta_{1}^{n} + \beta_{2}^{n})||u_{h}^{n}||_{0,t^{n}}^{2} + (1 + \Delta t \beta_{2}^{0})||u_{h}^{0}||_{0,t^{0}}^{2} + \Delta t \sum_{n=1}^{N+1} \left(\frac{2}{\mu} + \frac{\Delta t}{2}\right) ||f^{n+1/2}||_{0,t^{n+1/2}}^{2}. \end{split}$$

Finally, using Gronwall's lemma, we get

...

$$\begin{aligned} \|u_{h}^{N+1}\|_{0,t^{N+1}}^{2} &+ \frac{\Delta t}{4} \sum_{n=0}^{N} |||(u_{h}^{n+1} + u_{h}^{n})|||_{t^{n+1/2}}^{2} \\ &\leq \left((1 + \Delta t \beta_{2}^{0}) \|u_{h}^{0}\|_{0,t^{0}}^{2} + \Delta t \sum_{n=0}^{N} \left(\frac{2}{\mu} + \Delta t\right) \|f^{n+1/2}\|_{0,t^{n+1/2}}^{2} \right) \exp\left(\Delta t \sum_{n=1}^{N+1} \frac{\beta_{1}^{n} + \beta_{2}^{n}}{1 - \Delta t (\beta_{1}^{n} + \beta_{2}^{n})} \right) \end{aligned}$$

Note that the assumption on the time step in terms of the mesh velocity needs to be satisfied in order to get the stable ALE-SUPG solution. Hence, the Crank–Nicolson time discretization scheme is conditionally stable for the considered problem. \Box

5. Numerical results

To support the proposed conservative ALE-SUPG finite element scheme, numerical results are presented in this section. Two examples, (*i*) transient scalar equation with $\epsilon = 0.01$, $\mathbf{b} = \mathbf{0}$ and c = 0 in (1), and (*ii*) transient scalar equation with $\epsilon = 10^{-8}$, $\mathbf{b} = (1, 0)^T$ and c = 0 in (1), are considered. In both examples, the standard Galerkin solution and the SUPG solution are compared. In order to study the influence of the mesh velocity type convection and reaction terms introduced by the conservative ALE formulation (5) for different Δt and δ_K , the first example is considered with $\mathbf{b} = \mathbf{0}$ and c = 0. Further, the deformation of the domain in this example is given explicitly, and thus the restriction of the mesh velocity on the time step in Crank–Nicolson method, see lemma (4), can be studied. The second example is a convection dominated problem, and it is considered to show the applicability of the ALE-SUPG discretization to convection dominated problems in time-dependent domains. Further, the piecewise linear and piecewise quadratic finite elements are used for the spatial discretization in the first and second examples, respectively. Even though the second derivative in the SUPG formulation becomes zero for the linear finite elements, the influence will be negligible for a very small diffusive coefficient ϵ even in higher order finite element discretizations. Note that the SUPG method is needed only for problems with small diffusion coefficient. In computations, the SUPG parameter is chosen as

$$\delta_{K} = \begin{cases} \frac{\delta_{0}h_{K,t}}{\|\mathbf{b} - \mathbf{w}\|_{\infty}} & \text{if } \epsilon < h_{K,t}\|\mathbf{b} - \mathbf{w}\|_{\infty}, \\ 0 & \text{else,} \end{cases}$$

where δ_0 is a numerical parameter and $h_{K, t}$ is the time-dependent local cell size. Further, the resulting system of algebraic equation is solved using the direct solver UMFPACK.

5.1. Example 1

We consider the time-dependent equation (1) with $\epsilon = 0.01$, $\mathbf{b} = \mathbf{0}$, c = 0 and $u_0 = 1600 Y_1(1 - Y_1) Y_2(1 - Y_2)$. Even though the convection and the reaction terms are zero in this example, the mesh velocity introduces convection and reaction type terms through the conservative ALE formulation of the equation, see (5). Moreover, the effects of the mesh velocity on the numerical solution obtained with different time discretizations can clearly be studied by using this example



Fig. 1. L²-norm of the standard Galerkin solution obtained for different time-steps. (a): Implicit Euler, and (b): Crank-Nicolson.

since **b** = **0** and *c* = 0. For this example, let $\Omega_0 := (0, 1)^2$ be the initial (as well as reference) domain. Further, the Eulerian coordinate *x*(*Y*, *t*) $\in \Omega_t$ is given by

$$x(Y,t) = \mathcal{A}_t(Y) = \begin{cases} x_1 = Y_1(2 - \cos(20\pi t)) \\ x_2 = Y_2(2 - \cos(20\pi t)) \end{cases}$$

where $Y \in \Omega_0$. Note that the domain expands and contracts periodically with origin intact, but remains square. Now, the mesh velocity **w** becomes

$$\mathbf{w} = \frac{dY}{dt} = \frac{d}{dt} \left(\frac{X}{2 - \cos(20\pi t)} \right) = \left(\frac{20\pi x_1 \sin(20\pi t)}{2 - \cos(20\pi t)}, \frac{20\pi x_2 \sin(20\pi t)}{2 - \cos(20\pi t)} \right).$$

In computations, the piecewise linear interpolation in time is used to obtain the new position of the domain, i.e., $x_h(Y, \tau) \in \Omega_{\tau}$ for $\tau \in [t^n, t^{n+1}]$ is given by

$$x_h(Y,\tau) = \frac{\tau - t^n}{\Delta t} x_h(Y,t^{n+1}) + \frac{t^{n+1} - \tau}{\Delta t} x_h(Y,t^n).$$

Hence, the mesh velocity is obtained as

$$\mathbf{w}_h(Y,\tau) = \frac{x_h(Y,t^{n+1}) - x_h(Y,t^n)}{\Delta t}.$$

The triangulated unit square domain consists of 8192 triangles and using the piecewise linear finite element (P_1) on this mesh results in 4225 degrees of freedoms (DOFs). Further, the computations are performed until the dimensionless time T = 2.

We first present the numerical results obtained with the standard Galerkin discretization. The L^2 -norm of the solution obtained with different time-steps are presented in Fig. 1. Since both the modified implicit Euler and Crank-Nicolson schemes are stable, we expect a similar behavior in the computed solutions also. Though the L^2 -norm of the solutions obtained with both schemes are decreasing in general, it can be observed that the solution obtained with implicit Euler time discretization is more diffusive for higher values of Δt , see Fig. 1 (a). Nevertheless, the monotonicity of implicit Euler solution is not affected by the mesh velocity even for large values of Δt , which supports the stability estimate derived in lemma (2), as there is no time step (Δt) restriction with respect to the mesh velocity. Contrarily, the solution obtained with the Crank-Nicolson scheme shows high oscillations when the time step is large with respect to mesh velocity, and it confirms the stability estimates given in lemma (4).

We next present the numerical results obtained with the SUPG discretization. Since $\mathbf{b} = 0$, the SUPG parameter is calculated using

$$\delta_{K} = \begin{cases} \frac{\delta_{0}h_{K,t}}{\|\mathbf{w}\|_{\infty}} & \text{if} \quad \epsilon < h_{K,t}\|\mathbf{w}\|_{\infty}, \\ 0 & \text{else.} \end{cases}$$

The L^2 -norm of the standard Galerkin solution is compared with the SUPG solution obtained using the implicit Euler and Crank-Nicolson schemes in Fig. 2 and Fig. 3, respectively. For different values of δ_0 , the SUPG solution obtained with the



Fig. 2. L²-norm of the ALE-SUPG solution obtained with the implicit Euler method for different values of δ_0 . (a): $\Delta t = 0.01$, and (b): $\Delta t = 0.0001$.



Fig. 3. L^2 -norm of the ALE-SUPG solution obtained with Crank–Nicolson method for different values of δ_0 . (a): $\Delta t = 0.01$, and (b): $\Delta t = 0.0025$.

implicit Euler using $\Delta t = 0.01$ and $\Delta t = 0.0001$ are depicted in Fig. 2(a) and (b), respectively. Since the standard Galerkin solution obtained with $\Delta t = 0.01$ is already too diffusive, the effect of δ_0 is not visible in the SUPG solution for the considered values of δ_0 . However, the effect of δ_0 on the solution can clearly be observed in $\Delta t = 0.0001$ case, see Fig. 2(b). Further, the SUPG solution approaches the standard Galerkin solution when δ_0 decreases, which is obvious from the SUPG formulation.

Next, the Crank–Nicolson time discretization with $\Delta t = 0.01$ and 0.0025 is chosen to study the effects of δ_0 on the solution, see Fig. 3(a) and (b), respectively. Contrary to the implicit Euler case, the standard Galerkin solution obtained with the Crank–Nicolson time discretization is not too diffusive even in the large time step case, and hence the effects of δ_0 on the solution can clearly be seen even in $\Delta t = 0.01$ case, see Fig. 3(a). Interestingly, the amplitude of the oscillation in the L^2 -norm of the solution reduces when δ_0 increases, see Fig. 3(a).

Remark 2. The choice of stabilization parameter δ_K in SUPG scheme is of interest to many researchers. There are several proposals for δ_K in the literature, see for example [27–29] and the references therein. A comparative study on the choice of δ_K for the convection–diffusion problems in fixed domains can be found in [27]. However, the convective term varies in ALE formulations, and the optimal choice of δ_K for problems in time-dependent domains needs further investigation.



Fig. 4. Standard Galerkin solution of Example 2 at t = 10 with the implicit Euler method.



Fig. 5. The observed undershoots in the ALE-SUPG solution of Example 2 for different values of δ_0 . (a): Implicit Euler, and (b): Crank–Nicolson.

Remark 3. The numerical results presented in Figs. 2 and 3 show that the condition $\delta_K \leq \frac{\Delta t}{4}$ used to derive the stability is not strong. It indicates that the stability estimates could also be derived for scalar problems in time-dependent domains without this restriction as derived in [10] for fixed domains.

5.2. Example 2

We next consider an example that exemplifies a fluid-structure interaction problem. Define a time-dependent twodimensional channel

$$\Omega_t := \{(-3,9) \times (-3,3)\} \setminus \overline{\Omega}_t^S$$

that excludes a periodically oscillating (up and down) circular disc Ω_t^S , where the position of the disc, $x(Y, t) \in \Omega_t^S$, and the reference disc are given by

$$x(Y,t) = \mathcal{A}_t(Y) : \begin{cases} x_1 = Y_1 \\ x_2 = Y_2 + 0.5\sin(2\pi t/5), \end{cases} \text{ and } \Omega_0^S := \{(Y_1,Y_2) \in \mathbb{R}^2; \ Y_1^2 + Y_2^2 \le 1\}. \end{cases}$$

Further, define $\Gamma_D := \{-3\} \times (-3, 3)$ as the inflow boundary, and the remaining part $\Gamma_N := \partial \Omega_t \setminus \Gamma_D$ as Neumann boundary. We now solve the convection dominated transient scalar equation (1) with $\epsilon = 10^{-8}$, $\mathbf{b} = (1, 0)^T$ and c = 0. Further, we



Fig. 6. The observed overshoots in the ALE-SUPG solution of Example 2 for different values of δ_0 . (a): Implicit Euler, and (b): Crank–Nicolson.



Fig. 7. ALE-SUPG solution of Example 2 over the line y = 0 with different values of δ_0 . (a): Implicit Euler, and (b): Crank–Nicolson.

impose zero initial value, the homogeneous Neumann condition on Γ_N , and inhomogeneous Dirichlet condition

$$u_D(x_1, x_2) = \begin{cases} 1 & \text{on } \partial \Omega_t^S, \\ 0 & \text{on } \Gamma_D. \end{cases}$$

For the considered data, there will be a boundary layer on the upstream side of the oscillating circular disc, and two interior layers on the downstream side of the disc. Since the solid disc oscillates periodically, the position of the boundary and interior layers also change in time.

The channel is triangulated using the mesh generator "Triangle" [30,31], and it results in 9416 triangular cells and 19552 DOF with piecewise quadratic polynomial finite element (P_2) for the unknown u. The computations are performed until the dimensionless time T = 10 with the time step $\Delta t = 0.01$. Further, the linear elastic-solid update technique is used to handle the mesh movement that occurs due to the oscillations of the solid disc. At each time step, we first compute the displacement of the disc. We then solve the linear elastic equation in Ω_{t^n} to compute the inner points' displacement by considering the displacement on $\partial \Omega_{t^{n+1}}^S$ as the Dirichlet value. This elastic update technique avoids the remeshing during the entire computation, see [32] for more details on handling the mesh movement.

Since the solution u must be in the range of [0, 1] for the considered problem, any value of u which is below 0 is called undershoots and exceeds 1 is called overshoots. The standard Galerkin solution obtained with the implicit Euler method



Fig. 8. A sequence of ALE-SUPG solution of Example 2 obtained with $\delta_0 = 0.1$ and implicit Euler discretization at different instance t = 0.05, 4, 7, 10.

is presented in Fig. 4. As expected the solution contains spurious oscillations, and have more undershoots and overshoots. We next perform an array of computations with different values of δ_0 in SUPG discretization with $\Delta t = 0.01$ in both implicit Euler and Crank–Nicolson methods. The observed undershoots and the overshoots in the ALE-SUPG solution obtained with different values of δ_0 in implicit Euler and Crank–Nicolson methods are plotted in Fig. 5 and 6, respectively. The overshoots in the solution obtained with the Crank–Nicolson time discretization using $\delta_0 = 0.1$ and 0.5 are highly oscillatory, and therefore, a curve fitting plot is used to show the overshoots.

As it can be observed from the plots that the undershoots and overshoots in the solution obtained with the Euler method is less in comparison to the solution obtained with the Crank–Nicolson method (note that the scaling of figures is different). For both the Euler and Crank–Nicolson methods, the choice of $\delta_0 = 0.1$ among the considered values suppresses the undershoots and overshoots almost. The oscillations can further be suppressed by varying (increasing) δ_0 . Moreover, the plots of undershoots and overshoots provide only an indication for the choice of δ_0 to suppress the spurious oscillations in the numerical solution. However, the smearing effect of δ_0 on the numerical solution cannot be observed in Figs. 5 and 6. Therefore, to analyze the smearing effect, the obtained ALE-SUPG solution over the line y = 0 for different values of δ_0 at time t = 10 are plotted in Fig. 7. Further, the surface plot of the ALE-SUPG solution at different instances are plotted in Figs. 8 and 9 to present the smearing effect. In addition, the contour plots at time t = 10 for standard Galerkin with implicit Euler, SUPG with implicit Euler and SUPG with Crank–Nicolson discretizations are also depicted in Fig. 10. Even though, the SUPG discretization suppressed the spurious oscillations in the numerical solution almost, there are very small undershoots and



Fig. 9. A sequence of ALE-SUPG solution of Example 2 obtained with $\delta_0 = 0.1$ and Crank–Nicolson method at different instance t = 0.05, 4, 7, 10.

overshoots (the solution exceeds its range [0, 1] at most 10%) for the chosen $\delta_0 = 0.1$. We could reduce these undershoots and overshoots by increasing δ_0 , however, it will smear the solution further.

Remark 4. In this work, the inconsistent SUPG formulation is considered by neglecting the time derivative term in the stabilized formulation. The derivation of the stability estimates for the consistent SUPG formulation becomes challenging. Nevertheless, the numerical implementation of consistent SUPG form is not difficult at least for equations in stationary domains. However, the implementation of the consistent SUPG is not straightforward in time-dependent domains since

$$\sum_{K\in\mathcal{T}_{h,\tau}}\int_{K}\frac{du}{dt}\ \delta_{K}(\mathbf{b}-\mathbf{w}_{h})\cdot\nabla\nu\ dK\neq\sum_{K\in\mathcal{T}_{h,\tau}}\frac{d}{dt}\int_{K}u\ \delta_{K}(\mathbf{b}-\mathbf{w}_{h})\cdot\nabla\nu\ dK,$$

when $(\mathbf{b} - \mathbf{w}_h)$ is a function of time. It needs further investigation to handle this term in the analysis as well as in the implementation.

6. Summary

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In this work, a stabilized numerical scheme for a transient scalar equation in the time-dependent domain is proposed. In particular, a conservative ALE-SUPG finite element method is analyzed for a convection dominated transient equation in moving domains. The stability estimates of conservative ALE-SUPG finite element method with the backward Euler and Crank–Nicolson temporal discretizations are derived. It is shown that the stability of the semi-discrete (continuous in time) ALE-SUPG equation is independent of the mesh velocity, whereas the stability of the fully discrete problem in implicit Euler case is unconditionally stable and the Crank–Nicolson time discretization is only conditionally stable (time step depends on the mesh velocity). The proposed conservative ALE-SUPG scheme is demonstrated with appropriate examples. The influence of the SUPG parameter are studied, and the numerical results confirm the theoretical estimates.





Fig. 10. Contour plots of the solution of Example 2 obtained at time t = 10. (a): Standard Galerkin with implicit Euler, (b): SUPG with $\delta_0 = 0.1$ and implicit Euler, (c): SUPG with $\delta_0 = 0.1$ and Crank–Nicolson.

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