Parallel Sorting

Sathish Vadhiyar

Parallel Sorting Problem

- The input sequence of size N is distributed across P processors
- The output is such that
  - elements in each processor P_i is sorted
  - elements in P_i is greater than elements in P_{i-1} and lesser than elements in P_{i+1}

Parallel quick sort

- Naïve approach

- Start with a single processor; divide array into two sub-arrays
- Now involve one more processor
- Both the processors perform the next step of quick sort within their local subarrays
- And so on….till the number of subarrays equal the number of processors

- Disadvantage: Inefficient utilization of processors

Another algorithm

- This algorithm involves all the processors in all the iterations
- One of the processors, P0, begins by broadcasting one of its elements as the pivot element to all the processors
- Each processor then divides its local array into two sub-arrays
  - L_i: elements less than the pivot
  - G_i: elements greater than the pivot

Parallel Quick Sort

- Processors then divided into two groups:
  - First group will process the subsequent steps with L_i s
  - Second group with G_i s

- The sizes of the processor groups must be in the ratio of the number of elements in Ls and Gs to achieve load balance

- These number of elements can be found using an allreduce operation

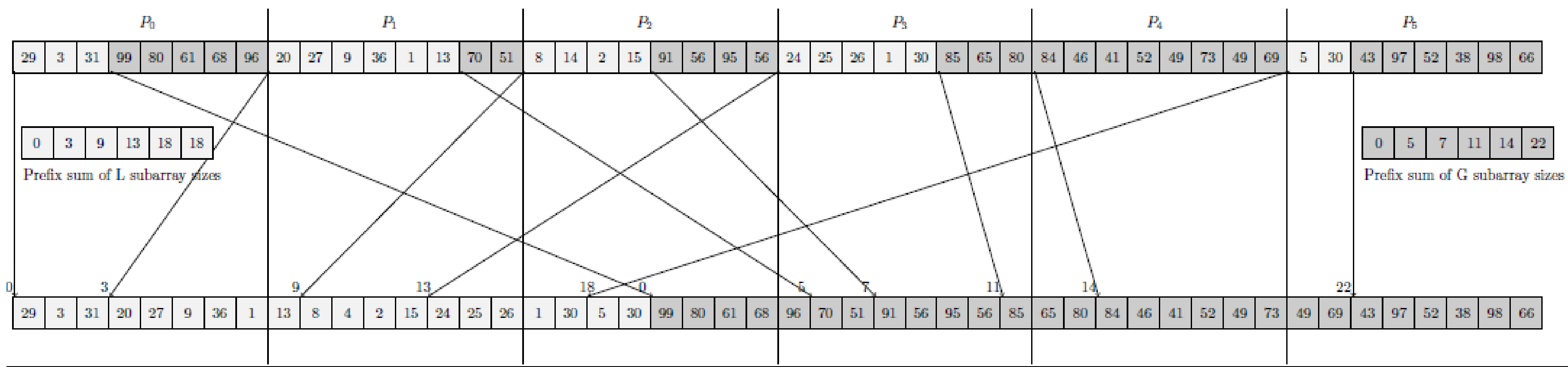
Shared memory implementation

- All L’s are formed in the first part of the array; all G’s in the second part
- Each processor needs to know the locations in the shared memory where it has to write its L_i and G_i
- Prefix sums of the sizes of the subarrays can be used

- Prefix sum can be done in O(logP)

Example: Prefix sum illustration

- In this example, 36 is the pivot element



Message Passing Version

- A processor should know which elements in its Li and Gi it should send to which processor
- Distributed prefix sum is used
- A processor can then deduce its destination processor for sending its L array using:
  - Total number of elements of L subarrays
  - prefix sums of sizes
  - Size of the processor group that will be responsible for L subarray
- Similarly for the G subarray
- In worst case, this requires all-to-all with time complexity O(N/P)

Parallel Quick sort

- The process now repeats with the subgroups
- Until the number of subgroups equal the number of processors
- At this stage, each processor performs a local quick sort: $O(N/P\log(N/P))$

Complexity and analysis

- log P times:
  - Broadcast: $O(\log P)$
  - Allreduce: $O(\log P)$
  - Prefix sum and all-to-all: $O(\log P + N/P)$
- Then local quick sort: $O(N/P.\log P)$

- Total: $O(N/P.\log(N/P)) + O(\log^2 P)+O(N/P.\log P)$

- Weaknesses: Load imbalance and under-utilization

Bitonic Sort

Bitonic sequence

- A sequence of length n is a bitonic sequence if
  - for an element i
    - elements a1<=a2<=a3<=….<=ai and
    - Elements ai >= ai-1 >= ai-2>=…>=an
  - Any cyclic rotation of such a sequence is also a bitonic sequence

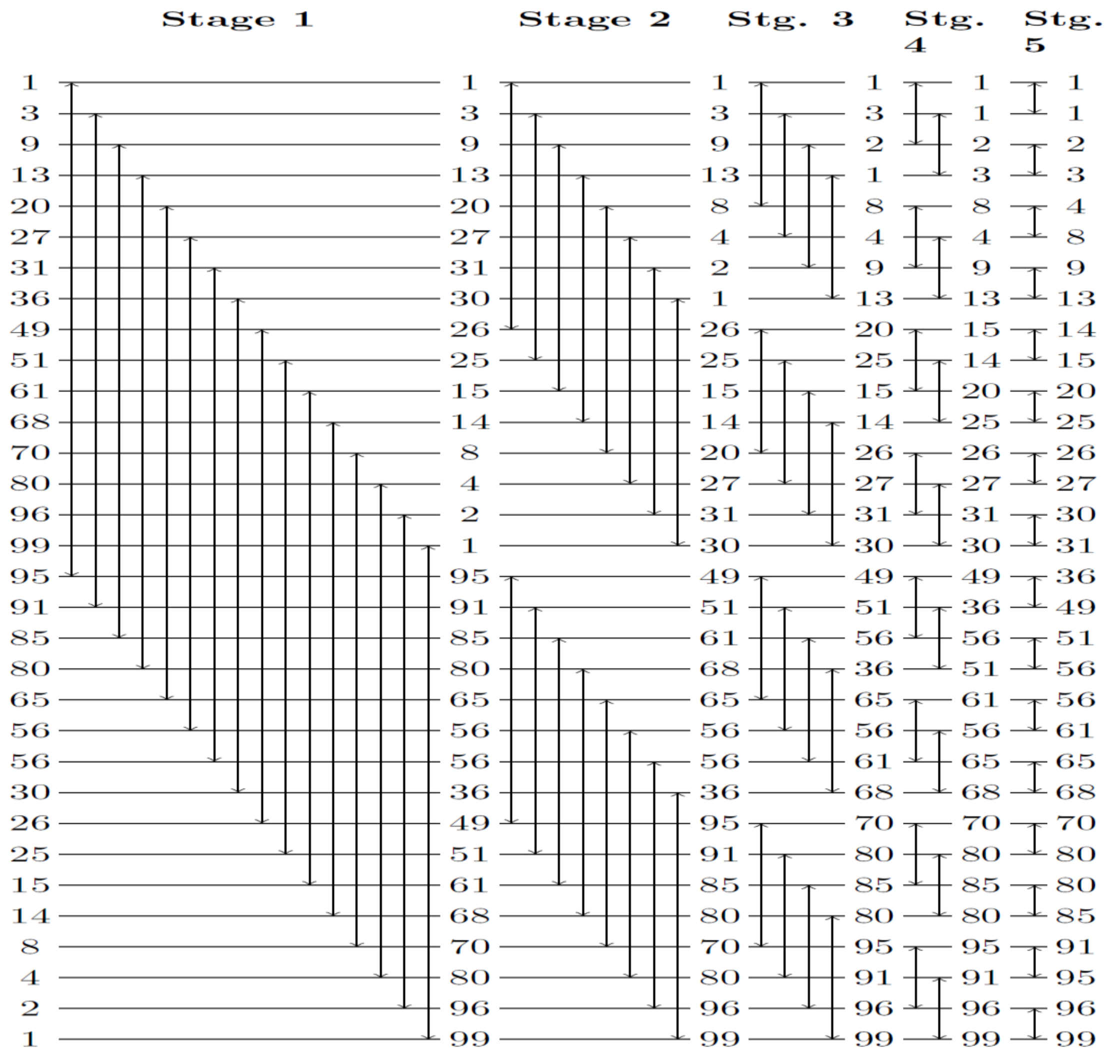
Bitonic property

- Given a bitonic sequence A, let us form another sequence B such that:
  - B[i] = min(A[i],A[i+N/2])
  - B[i+N/2] = max(A[i],A[i+N/2])

- It is easy to prove that:
  - Lower half B[0]….B[N/2-1] <= upper half B[N/2]…B[N-1]
  - Both the halves themselves are bitonic sequences of lengths N/2

Converting bitonic sequence into a sorted sequence

- To convert bitonic sequence of length N into a sorted sequence, we repeat the above recursively:
- In the first stage, form two bitonic sub-sequences of N/2 each
- In the second stage, form four bitonic sub-sequences of N/4 each
- ….
- After logN stages, a sorted sequence is formed
- This process is called **bitonic merge**


Stage 1
Stage 2
Stg. 3
Stg. 4
Stg. 5

Bitonic sort

- Convert the original unsorted sequence into a bitonic sequence, then use the above procedure to convert to a sorted sequence

- Converting unsorted sequence of length N into a bitonic sequence of length N:
- Larger and larger bitonic sequences are built starting from sequences of lengths 2
- Note that any sequence of length 2 is a bitonic sequence

Bitonic sort

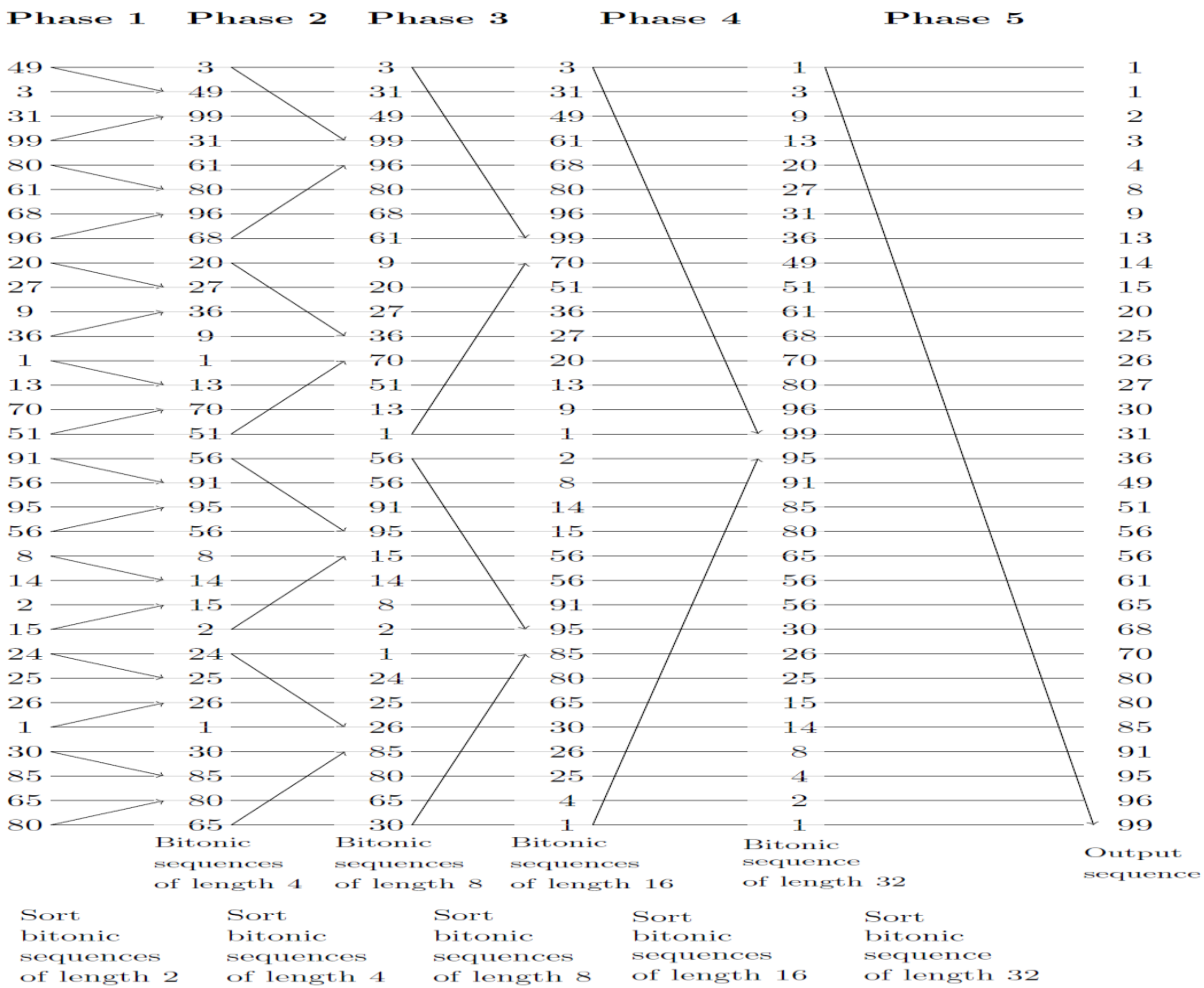
- In the first phase:
- Sort two consecutive sub-sequences of lengths 2 such that
  - the first subsequence is sorted in ascending order, second in descending order
- Now the two sorted sub-sequences are merged to form a bitonic sequence of length 4.

- In the second phase:
  - Consider two consecutive sub-sequences of lengths 4
  - Sort them into ascending and descending
  - Merge them into bitonic sequence of length 8

Bitonic sort

- So on….
- At the end of logN phases, a bitonic sequence of length N formed, which is converted into a sorted sequence

- Time complexity:
- logN phases
- Phase i has i stages
- $O(\log^2 N)$


Phase 1
Phase 2
Phase 3
Phase 4
Phase 5
49 3 31 99 80 61 68 96 20 27 9 36 1 13 70 51 91 56 95 56 8 14 2 15 24 25 26 1 30 85 65 80
3 49 99 31 61 80 96 68 20 27 36 9 1 13 70 51 56 91 95 56 8 14 15 2 24 25 26 1 30 85 80 65
3 31 49 99 96 80 68 61 9 20 27 36 70 51 13 1 56 56 91 95 15 14 8 2 1 24 25 26 85 80 65 30
3 31 49 61 68 80 96 99 70 51 36 27 20 13 9 1 2 8 14 15 56 56 91 95 85 80 65 30 26 25 4 1
1 3 9 13 20 27 31 36 49 51 61 68 70 80 96 99 95 91 85 80 65 56 56 30 26 25 15 14 8 4 2 1
1 1 2 3 4 8 9 13 14 15 20 25 26 27 30 31 36 49 51 56 56 61 65 68 70 80 80 85 91 95 96 99
Bitonic sequences of length 4
Bitonic sequences of length 8
Bitonic sequences of length 16
Bitonic sequence of length 32
Output sequence
Sort bitonic sequences of length 2
Sort bitonic sequences of length 4
Sort bitonic sequences of length 8
Sort bitonic sequences of length 16
Sort bitonic sequence of length 32

Sequential complexity

- Has logN phases
- Each phase i has i stages
- Each stage i performs N compare-exchange operations
- Hence $O(N\log^2 N)$

Parallelization
Hypercube and mesh networks

- Maps well to hypercubes
- Processors are mapped to corresponding hypercube nodes
- Processors that need to interact for compare-exchange operations in the phases are mapped to hypercube nodes that have direct connections
- For mesh networks, a shuffle-row mapping is used

Parallel implementation General networks

- Array distributed into block distribution across P processors
- The last logP of the logN phases require communications for exchanging elements
- In the last phase, out of the logN stages, the first logP stages involve communications
- Each communication is a compare-and-exchange
- Hence $O(\log^2 P)$ communication steps

- $O(N/P.\log^2 P)$ communications
- $O(N/P\log^2 N)$ computations

Observations

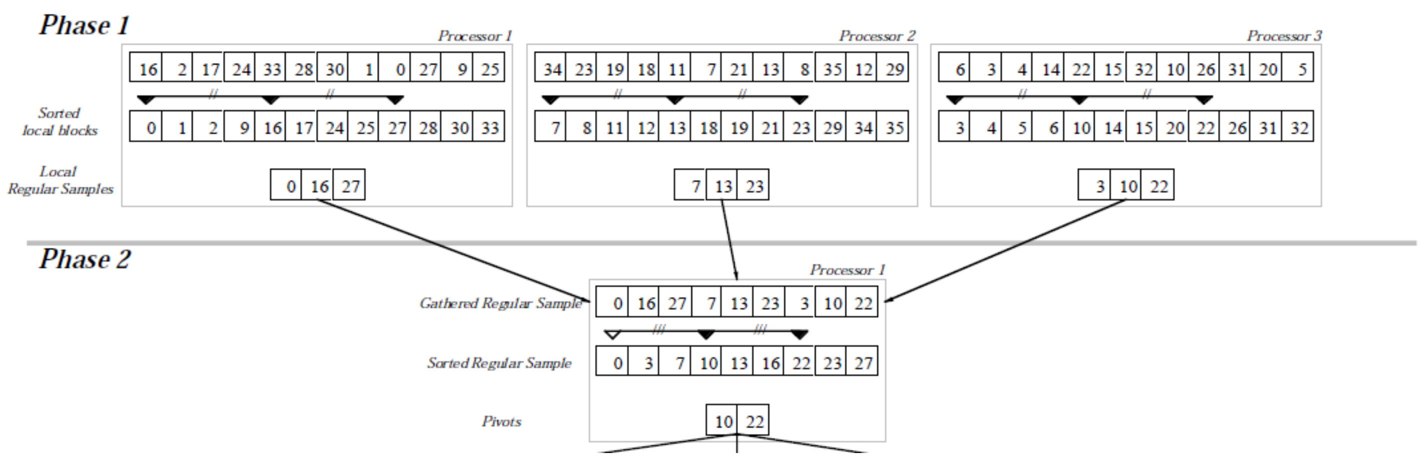
- In general, applied to small sequences due to high computation complexity
- Has poor speedup for greater than thousand processors due to high communication complexities

- Sample Sort

Parallel Sorting by Regular Sampling (PSRS)

1. Each processor sorts its local data
2. Each processor selects a sample vector of size p-1; kth element is (n/p * (k+1)/p)
3. Samples are sent and merge-sorted on processor 0
4. Processor 0 defines a vector of p-1 *splitters* starting from p/2 element; i.e., kth element is p(k+1/2); broadcasts to the other processors

Example



Phase 1
Processor 1
16 2 17 24 33 28 30 1 0 27 9 25
Sorted local blocks
0 1 2 9 16 17 24 25 27 28 30 33
Local Regular Samples
0 16 27
Processor 2
34 23 19 18 11 7 21 13 8 35 12 29
7 8 11 12 13 18 19 21 23 29 34 35
7 13 23
Processor 3
6 3 4 14 22 15 32 10 26 31 20 5
3 4 5 6 10 14 15 20 22 26 31 32
3 10 22
Phase 2
Processor 1
Gathered Regular Sample
0 16 27 7 13 23 3 10 22
Sorted Regular Sample
0 3 7 10 13 16 22 23 27
Pivots
10 22

PSRS

5. Each processor sends local data to correct destination processors based on splitters; all-to-all exchange
6. Each processor merges the data chunk it receives

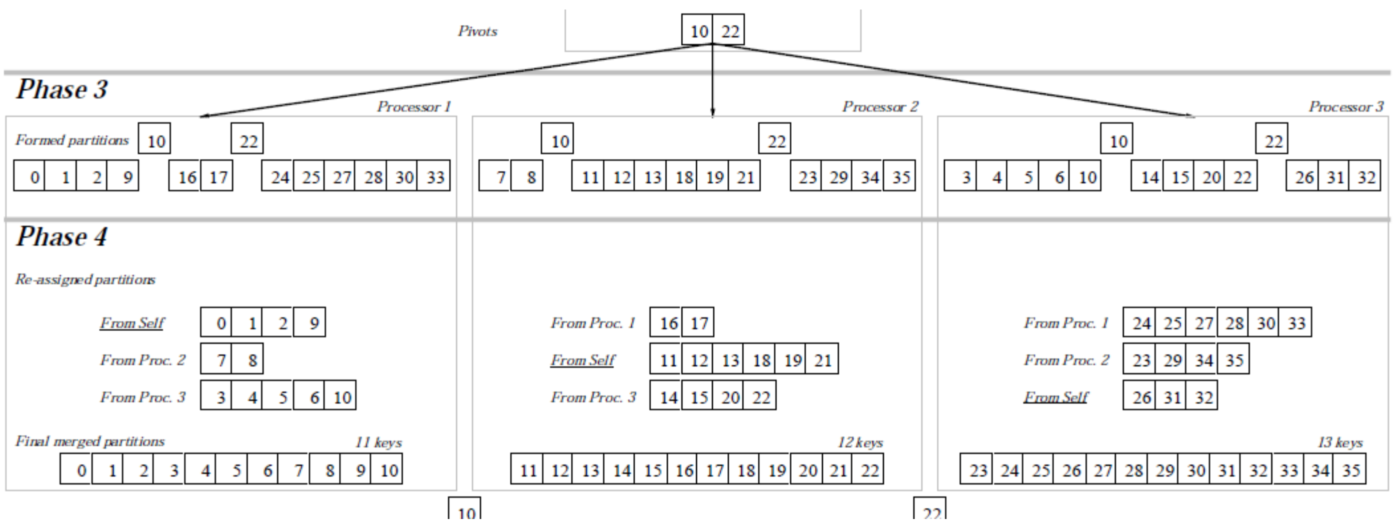
Step 5

- Each processor finds where each of the p-1 pivots divides its list, using a binary search
- i.e., finds the index of the largest element number larger than the jth pivot
- At this point, each processor has p sorted sublists with the property that each element in sublist i is greater than each element in sublist i-1 in any processor

Step 6

- Each processor i performs a p-way merge-sort to merge the ith sublists of p processors

Example Continued



Analysis

- The first phase of local sorting takes $O((n/p)\log(n/p))$
- 2nd phase:
  - Sorting $p(p-1)$ elements in processor 0 – $O(p^2 \log p^2)$
  - Each processor performs p-1 binary searches of n/p elements – $p\log(n/p)$
- 3rd phase: Each processor merges (p-1) sublists
  - Size of data merged by any processor is no more than 2n/p (proof)
  - Complexity of this merge sort $2(n/p)\log p$
- Summing up: $O((n/p)\log n)$

Analysis

- 1st phase – no communication
- 2nd phase – p(p-1) data collected; p-1 data broadcast
- 3rd phase: Each processor sends (p-1) sublists to other p-1 processors; processors work on the sublists independently

Analysis

Not scalable for large number of processors

Merging of p(p-1) elements done on one processor; 16384 processors require 16 GB memory

Sorting by Random Sampling

- An interesting alternative; random sample is flexible in size and collected randomly from each processor's local data
- Advantage
  - A random sampling can be retrieved before local sorting; overlap between sorting and splitter calculation

Radix Sort

- During every step, the algorithm puts every key in a bucket corresponding to the value of some subset of the key's bits
- A k-bit radix sort looks at k bits every iteration
- Easy to parallelize – assign some subset of buckets to each processor
- Load balance – assign variable number of buckets to each processor

Radix Sort – Load Balancing

- Each processor counts how many of its keys will go to each bucket
- Sum up these histograms with reductions
- Once a processor receives this combined histogram, it can adaptively assign buckets

Radix Sort - Analysis

- Requires multiple iterations of costly all-to-all
- Cache efficiency is low – any given key can move to any bucket irrespective of the destination of the previously indexed key
- Affects communication as well

Histogram Sort

- Another splitter-based method
- Histogram also determines a set of p-1 splitters
- It achieves this task by taking an iterative approach rather than one big sample
- A processor broadcasts k (> p-1) initial splitter guesses called a probe
- The initial guesses are spaced evenly over data range

Histogram Sort Steps

1. Each processor sorts local data
2. Creates a histogram based on local data and splitter guesses
3. Reduction sums up histograms
4. A processor analyzes which splitter guesses were satisfactory (in terms of load)
5. If unsatisfactory splitters, the , processor broadcasts a new probe, go to step 2; else proceed to next steps

Histogram Sort Steps

6. Each processor sends local data to appropriate processors – all-to-all exchange
7. Each processor merges the data chunk it receives

Merits:

- Only moves the actual data once
- Deals with uneven distributions

Sources/References

- On the versatility of parallel sorting by regular sampling. Li et al. Parallel Computing. 1993.
- Parallel Sorting by regular sampling. Shi and Schaeffer. JPDC 1992.
- Highly scalable parallel sorting. Solomonic and Kale. IPDPS 2010.